In Search of the Perfect Fluid

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Fluids: Gases, liquids, plasmas, …

Hydrodynamics: Long-wavelength, low-frequency dynamics of conserved or spontaneously broken symmetry variables.

Historically: Water

$$(\rho, \epsilon, \vec{\pi})$$
Simple non-relativistic fluid

Simple fluid: Conservation laws for mass, energy, momentum

\[
\frac{\partial \rho}{\partial t} + \vec{\nabla} (\rho \vec{v}) = 0
\]

\[
\frac{\partial \epsilon}{\partial t} + \vec{\nabla} j^\epsilon = 0
\]

\[
\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0
\]

Constitutive relations: Energy momentum tensor

\[
\Pi_{ij} = P \delta_{ij} + \rho v_i v_j + \eta \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2)
\]

reactive  dissipative  2nd order

Expansion \( \Pi_{ij}^0 \gg \delta \Pi_{ij}^1 \gg \delta \Pi_{ij}^2 \)
Regime of applicability

Expansion parameter $Re^{-1} = \frac{\eta(\partial v)}{\rho v^2} = \frac{\eta}{\rho L v} \ll 1$

$$Re = \frac{\hbar n}{\eta} \times \frac{mvL}{\hbar}$$

fluid property  flow property

Kinetic theory estimate: $\eta \sim npl_{mfp}$

$$Re^{-1} = \frac{v}{c_s} Kn \quad Kn = \frac{l_{mfp}}{L}$$

expansion parameter $Kn \ll 1$
Shear viscosity

Viscosity determines shear stress ("friction") in fluid flow

\[ F = A \eta \frac{\partial v_x}{\partial y} \]

Kinetic theory: conserved quantities carried by quasi-particles

\[ \frac{\partial f_p}{\partial t} + \vec{v} \cdot \nabla_x f_p + \vec{F} \cdot \nabla_p f_p = C[f_p] \]

\[ \eta \sim \frac{1}{3} n \bar{p} l_{mf_p} \]

Dilute, weakly interacting gas: \( l_{mf_p} \sim 1/(n\sigma) \)

\[ \eta \sim \frac{1}{3} \frac{\bar{p}}{\sigma} \] independent of density!
Shear viscosity

non-interacting gas ($\sigma \to 0$):

$$\eta \to \infty$$

non-interacting and hydro limit ($T \to \infty$) limit do not commute

strongly interacting gas:

$$\frac{\eta}{n} \sim \overline{p} l_{mfp} \geq \hbar$$

but: kinetic theory not reliable!

what happens if the gas condenses into a liquid?

Eyring, Frenkel:

$$\eta \simeq hn \exp(E/T) \geq hn$$
And now for something completely different . . .

**String Theory Summarized:**

I just had an awesome idea. Suppose all matter and energy is made of tiny, vibrating "strings."

Okay, what would that imply?

I dunno.
Gauge theory at strong coupling: Holographic duality

The AdS/CFT duality relates

large $N_c$ (conformal) gauge theory in 4 dimensions ⇔ string theory on 5 dimensional Anti-de Sitter space $\times S_5$
correlation fcts of gauge invariant operators ⇔ boundary correlation fcts of AdS fields

$$\langle \exp \int dx \, \phi_0 \mathcal{O} \rangle = Z_{string}[\phi(\partial AdS) = \phi_0]$$

The correspondence is simplest at strong coupling $g^2 N_c$

strongly coupled gauge theory ⇔ classical string theory
Holographic duals: Transport properties

Thermal (conformal) field theory $\equiv \text{AdS}_5$ black hole

CFT temperature $\Leftrightarrow$ Hawking temperature

CFT entropy $\Leftrightarrow$ Hawking-Bekenstein entropy

$\sim$ area of event horizon

Shear viscosity $\Leftrightarrow$ Graviton absorption cross section

$\sim$ area of event horizon

\begin{align*}
T_{\mu\nu} &= \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \\
g_{\mu\nu} &= g_{\mu\nu}^0 + \gamma_{\mu\nu}
\end{align*}
Holographic duals: Transport properties

Thermal (conformal) field theory $\equiv AdS_5$ black hole

- CFT entropy $\Leftrightarrow$ Hawking-Bekenstein entropy
  $\sim$ area of event horizon

- shear viscosity $\Leftrightarrow$ Graviton absorption cross section
  $\sim$ area of event horizon

Strong coupling limit

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}$$

Son and Starinets (2001)

Strong coupling limit universal? Provides lower bound for all theories?
**Kinetics vs no-kinetics**

AdS/CFT low viscosity goo

pQCD kinetic plasma
Effective theories for fluids (Here: Weak coupling QCD)

\[ \mathcal{L} = \bar{q}_f (iD - m_f) q_f - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \]

\[ \frac{\partial f_p}{\partial t} + \vec{v} \cdot \nabla_x f_p = C[f_p] \quad (\omega < T) \]

\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < g^4 T) \]
Effective theories (Strong coupling)

\[ \mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4} G^a_{\mu\nu}G^a_{\mu\nu} + \ldots \Leftrightarrow S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \mathcal{R} + \ldots \]

\[ \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < T) \]
Kinetics vs no-kinetics

Spectral function \( \rho(\omega) = \text{Im} G_R(\omega, 0) \) associated with \( T_{xy} \)

weak coupling QCD  \hspace{2cm} \text{strong coupling AdS/CFT}

transport peak vs no transport peak
Perfect Fluids: How to be a contender?

Bound is quantum mechanical

need quantum fluids

Bound is incompatible with weak coupling and kinetic theory

strong interactions, no quasi-particles

Model system has conformal invariance (essential?)

(Almost) scale invariant systems
Perfect Fluids: The contenders

QGP \( (T=180 \text{ MeV}) \)

Trapped Atoms \( (T=0.1 \text{ neV}) \)

Liquid Helium \( (T=0.1 \text{ meV}) \)
Perfect Fluids: The contenders

QGP $\eta = 5 \cdot 10^{11} \text{Pa} \cdot \text{s}$

Trapped Atoms $\eta = 1.7 \cdot 10^{-15} \text{Pa} \cdot \text{s}$

Liquid Helium $\eta = 1.7 \cdot 10^{-6} \text{Pa} \cdot \text{s}$

Consider ratios $\eta/s$
QCD and the Quark Gluon Plasma

$$\mathcal{L} = \bar{q}_f(i\slashed{D} - m_f)q_f - \frac{1}{4g^2}G^a_{\mu\nu}G^a_{\mu\nu}$$
Quantum chromodynamics (QCD)

Elementary fields:

**Quarks**
- Color: \( a = 1, \ldots, 3 \)
- Spin: \( \alpha = 1, 2 \)
- Flavor: \( f = u, d, s, c, b, t \)

**Gluons**
- Color: \( a = 1, \ldots, 8 \)
- Spin: \( \epsilon^\pm_\mu \)

Dynamics: Generalized Maxwell (Yang-Mills) + Dirac theory

\[
\mathcal{L} = \bar{q}_f (i \slashed{D} - m_f) q_f - \frac{1}{4} G^a_{\mu \nu} G^{a \mu \nu}
\]

\[
G^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu
\]

\[
i \slashed{D} q = \gamma^\mu \left( i \partial_\mu + g A^a_\mu t^a \right) q
\]
Asymptotic freedom

Modification of Coulomb interaction due to quantum fluctuations

$q\bar{q}$-pairs  electric gluons  magnetic gluons

\[ A_{\mu}^{cl} \quad \delta \psi \quad A_{\nu}^{cl} \quad \delta A_{\mu} \quad k^\nu F_{\mu\nu} \]

\begin{align*}
\text{dielectric } & \epsilon > 1 \quad \text{dielectric } \epsilon > 1 \quad \text{paramagnetic } \mu > 1 \\
\mu \epsilon &= 1 \ \Rightarrow \ \epsilon < 1
\end{align*}

\[
\beta(g) = -\frac{\partial g}{\partial \log(r)} = \frac{g^3}{(4\pi)^2} \left\{ \left[ \frac{1}{3} - 4 \right] N_c + \frac{2}{3} N_f \right\} < 0
\]
“Seeing” quarks and gluons
Running coupling constant

\[ \beta(g) \alpha_s - \left( \frac{11g}{3} \right) \alpha_s^2 \]

\[ \alpha_s(Q) \]

\[ \alpha_s(M_z) = 0.118 \pm 0.003 \]

\[ Q/\text{[GeV]} \]
The high T phase: Qualitative argument

High T phase: Weakly interacting gas of quarks and gluons?

**typical momenta** \( p \sim 3T \)

Large angle scattering involves large momentum transfer

**effective coupling is small**

Small angle scattering is screened (not anti-screened!)

**coupling does not become large**

Quark Gluon Plasma
Dilute Fermi gas: BCS-BEC crossover

\[ \mathcal{L}_{\text{eff}} = \psi^\dagger \left( i \partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 \]
Consider simple square well potential

\[
a < 0 \quad a = \infty, \epsilon_B = 0 \quad a > 0, \epsilon_B > 0
\]
Unitarity limit

Now take the range to zero, keeping $\epsilon_B \simeq 0$

Universal relations

$$\mathcal{T} = \frac{1}{ik + 1/a}$$

$$\epsilon_B = \frac{1}{2ma^2}$$

$$\psi_B \sim \frac{1}{\sqrt{ar}} \exp(-r/a)$$
Feshbach resonances

Atomic gas with two spin states: “↑” and “↓”

Feshbach resonance

\[ a(B) = a_0 \left( 1 + \frac{\Delta}{B - B_0} \right) \]

“Unitarity” limit \( a \to \infty \)

\[ \sigma = \frac{4\pi}{k^2} \]
Almost ideal fluid dynamics (cold gases)

Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy

O’Hara et al. (2002)
Collective oscillations

Radial breathing mode

Ideal fluid hydrodynamics \((P = \frac{2}{3} \mathcal{E})\)

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = 0
\]

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\nabla P \frac{1}{mn} - \nabla V \frac{1}{m}
\]

Hydro frequency at unitarity

\[
\omega = \sqrt{\frac{10}{3}} \omega_{\perp}
\]

Damping small, depends on \(T/T_F\).

experiment: Kinast et al. (2005)
Viscous hydrodynamics

Energy dissipation ($\eta, \zeta, \kappa$: shear, bulk viscosity, heat conductivity)

\[
\dot{E} = -\frac{1}{2} \int d^3 x \eta(x) \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right)^2 \\
- \int d^3 x \zeta(x) (\partial_i v_i)^2 - \frac{1}{T} \int d^3 x \kappa(x) (\partial_i T)^2
\]

Shear viscosity to entropy ratio
(assuming $\zeta = \kappa = 0$)

\[
\frac{\eta}{s} = (3\lambda N)^{\frac{1}{3}} \frac{\Gamma}{\omega_\perp} \frac{E_0}{E_F} \frac{N}{S}
\]

Schaefer (2007), see also Bruun, Smith

$T \ll T_F$ \hspace{1cm} $T \gg T_F, \tau_R \simeq \eta/P$
Dissipation

Dissipation

\[
\frac{(\delta t_0)}{t_0} = \begin{cases} 
0.008 \\
0.024 
\end{cases} 
\left( \frac{\langle \eta/s \rangle}{1/(4\pi)} \right) \left( \frac{2 \cdot 10^5}{N} \right)^{1/3} \left( \frac{S/N}{2.3} \right) \left( \frac{0.85}{E_0/E_F} \right)
\]

- \(t_0\): "Crossing time" (\(b_\perp = b_z\), \(\theta = 45^\circ\))
- \(a\): amplitude
Elliptic flow (QGP)

Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy.

\[
p_0 \left. \frac{dN}{d^3 p} \right|_{p_z=0} = v_0(p_{\perp}) (1 + 2v_2(p_{\perp}) \cos(2\phi) + \ldots)
\]

Elliptic flow: initial entropy scaling

Viscosity and elliptic flow

Consistency condition $T_{\mu\nu} \gg \delta T_{\mu\nu}$ (applicability of Navier-Stokes)

$$\frac{\eta + \frac{4}{3} \zeta}{s} \ll \frac{3}{4} (\tau T)$$

Danielewicz, Gyulassy (1985)

Very restrictive for $\tau < 1$ fm

Many questions: Dependence on initial conditions, freeze out, etc.

conservative bound $\frac{\eta}{s} < 0.4$
The bottom-line

Remarkably, the best fluids that have been observed are the coldest and the hottest fluid ever created in the laboratory, cold atomic gases \((10^{-6}\text{K})\) and the quark gluon plasma \((10^{12}\text{K})\) at RHIC.

Both of these fluids come close to a bound on the shear viscosity that was first proposed based on calculations in string theory, involving non-equilibrium evolution of back holes in 5 (and more) dimensions.
Extra Slides
## Kinetic theory: Quasiparticles

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Theory Summary

- Unitary gas
- $^4\text{He}$
- QCD
Spectral function (lattice QCD)

\[ \rho(\omega) K(x_0=1/2T,\omega)/T^4 \]

<table>
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<th>( T )</th>
<th>1.02 ( T_c )</th>
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<td>( \eta/s )</td>
<td>0.102(56)</td>
<td>0.134(33)</td>
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<tr>
<td>( \zeta/s )</td>
<td>0.73(3)</td>
<td>0.065(17)</td>
<td>0.008(7)</td>
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H. Meyer (2007)
Experiment (liquid helium)

Kapitza (1938)
viscosity vanishes below $T_c$
capillary flow viscometer

Hollis-Hallett (1955)
roton minimum, phonon rise
rotation viscometer

$$\frac{\eta}{s} \simeq 0.8 \frac{\hbar}{k_B}$$
Time scales

\( R_i \) [\( \mu m \)]

\( t_{\text{acc}} \), \( t_{\text{diss}} \), \( t_{\text{fr}} \), \( t_{\text{cross}} \)

\( R_\perp \)

\( R_z \)

\( t[ms] \)