Strongly interacting quantum fluids: Transport theory

Thomas Schaefer

North Carolina State University
Fluids: Gases, Liquids, Plasmas, …

Hydrodynamics: Long-wavelength, low-frequency dynamics of conserved or spontaneously broken symmetry variables.

\[ \tau \sim \tau_{micro} \]

Historically: Water

\((\rho, \epsilon, \vec{\pi})\)
Simple non-relativistic fluid

Simple fluid: Conservation laws for mass, energy, momentum

\[ \frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0 \]

\[ \frac{\partial \varepsilon}{\partial t} + \vec{\nabla} j^\varepsilon = 0 \]

\[ \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \]

Constitutive relations: Energy momentum tensor

\[ \Pi_{ij} = P \delta_{ij} + \rho v_i v_j + \eta \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2) \]

reactive \quad dissipative \quad 2nd order

Expansion \( \Pi_{ij}^0 \gg \delta \Pi_{ij}^1 \gg \delta \Pi_{ij}^2 \)
Regime of applicability

Expansion parameter

\[ \text{Re}^{-1} = \frac{\eta(\partial v)}{\rho v^2} = \frac{\eta}{\rho L v} \ll 1 \]

\[ \text{Re} = \frac{\hbar n}{\eta} \times \frac{mvL}{\hbar} \]

fluid property  \quad flow property

Kinetic theory estimate: \( \eta \sim npl_{mfp} \)

\[ \text{Re}^{-1} = \frac{v}{c_s} Kn \quad Kn = \frac{l_{mfp}}{L} \]

expansion parameter \( Kn \ll 1 \)
Relativistic hydrodynamics

Energy momentum tensor of an ideal fluid

\[ T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu}, \]

Energy-momentum conservation: \( \partial_\mu T^{\mu\nu} = 0 \)

\[ \partial_\mu (su^\mu) = 0 \quad \text{and} \quad Du_\mu = -\frac{1}{\epsilon + P}\nabla_{\mu} \nabla_{\nu} P \]

\[ D = u \cdot \partial \quad \nabla_{\mu} = \Delta_{\mu\nu} \partial^\nu \quad \text{and} \quad \Delta_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu \]

Viscous contribution

\[ \delta^{(1)} T^{\mu\nu} = -\eta\sigma^{\mu\nu} - \zeta \Delta^{\mu\nu} \partial \cdot u \]

\[ \sigma^{\mu\nu} = \Delta^{\mu\alpha} \Delta^{\nu\beta} \left( \partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} \eta_{\alpha\beta} \partial \cdot u \right). \]
Relativistic hydrodynamics: causality

Linearized hydro: consider small fluctuations \( g^i = \delta T^{0i} \)

\[
S^L_{gg} = 2sT \frac{\Gamma_s \omega^2 k^2}{(\omega^2 - c_s^2 k^2)^2 + (\Gamma_s \omega k^2)^2}
\]

\[
S^T_{gg} = \frac{2\eta k^2}{\omega^2 + (\frac{\eta}{sT} k^2)^2}
\]

\[
\Gamma_s = \frac{4}{3} \frac{\eta + \zeta}{sT}
\]

L/T channel: propagating sound mode, diffusive shear mode.

Consider “speed” of shear wave

\[
v_{diff} = \frac{\partial |\omega|}{\partial k} = \frac{\eta}{sT} k
\]

Find acausal* behavior \( v_{diff} > c \) for \( k > k_{cr} \).

* Occurs outside regime of validity of hydro. But: Causes numerical difficulties.
Second order hydrodynamics

Causality can be restored by introducing a finite relaxation time

$$\eta(\omega) \simeq \frac{\eta}{1 + i\omega\tau_\pi}$$

More formal approach: Second order hydrodynamics (BRSSS)

$$\delta^{(2)} T^{\mu \nu} = \eta \tau_{II} \left[ \langle D\sigma^{\mu \nu} \rangle + \frac{1}{3} \sigma^{\mu \nu} (\partial \cdot u) \right]$$

$$+ \lambda_1 \sigma^{\langle \mu \lambda \sigma^\nu \rangle \lambda} + \lambda_2 \sigma^{\langle \mu \lambda \Omega^\nu \rangle \lambda} + \lambda_3 \Omega^{\langle \mu \lambda \Omega^\nu \rangle \lambda}$$

$$A^{\langle \mu \nu \rangle} = \frac{1}{2} \Delta^{\mu \alpha} \Delta^{\nu \beta} \left( A_{\alpha \beta} + A_{\beta \alpha} - \frac{2}{3} \Delta^{\mu \nu} \Delta^{\alpha \beta} A_{\alpha \beta} \right), \quad \Omega^{\mu \nu} = \frac{1}{2} \Delta^{\mu \alpha} \Delta^{\nu \beta} \left( \partial_\alpha u_\beta - \partial_\beta u_\alpha \right)$$

Contains four new transport coefficients $\tau_{II}, \lambda_i$

Can be written as a relaxation equation for $\pi^{\mu \nu} \equiv \delta T^{\mu \nu}$

$$\pi^{\mu \nu} = -\eta \sigma^{\mu \nu} - \tau_{II} \langle D\pi^{\mu \nu} \rangle + \ldots$$
Shear viscosity

Viscosity determines shear stress ("friction") in fluid flow

\[ F = A \eta \frac{\partial v_x}{\partial y} \]

Kinetic theory: conserved quantities carried by quasi-particles

\[
\frac{\partial f_p}{\partial t} + \bar{v} \cdot \bar{\nabla}_x f_p + \bar{F} \cdot \bar{\nabla}_p f_p = C[f_p]
\]

\[ \eta \sim \frac{1}{3} n \bar{p} l_{mfp} \]

Dilute, weakly interacting gas: \( l_{mfp} \sim 1/(n\sigma) \)

\[ \eta \sim \frac{1}{3} \frac{\bar{p}}{\sigma} \quad \text{independent of density!} \]
Shear viscosity

non-interacting gas \((\sigma \to 0)\): \(\eta \to \infty\)

non-interacting and hydro limit \((T \to \infty)\) limit do not commute

strongly interacting gas:
\[
\frac{\eta}{n} \sim \bar{p}l_{mfp} \geq \hbar
\]

but: kinetic theory not reliable!

what happens if the gas condenses into a liquid?

[Graph showing pair correlation function for liquid and gas]
Viscosity of a liquid is a dominant theme here, and you know Vicki’s program of explaining everything in terms of fundamental constants. The viscosity of a liquid is a tough nut to crack . . . because when the stuff is cooled by $40^\circ$ its viscosity can change by a factor of $10^6$.

Purcell anticipates

Vicki’s theory

$$\eta \sim \exp\left(\frac{E}{T}\right)$$

But it’s more mysterious than that: The viscosities have a big range, but they all stop in the same place. I don’t understand that.

Eyring, Frenkel:

$$\eta \simeq h n \exp\left(\frac{E}{T}\right)$$
And now for something completely different . . .

STRING THEORY SUMMARIZED:

I just had an awesome idea. Suppose all matter and energy is made of tiny, vibrating "strings."

Okay. What would that imply?

I dunno.
Gauge theory at strong coupling: Holographic duality

The AdS/CFT duality relates

large $N_c$ (conformal) gauge theory in 4 dimensions ⇔ string theory on 5 dimensional Anti-de Sitter space $\times S_5$
correlation fcts of gauge invariant operators ⇔ boundary correlation fcts of AdS fields

$$\langle \exp \int dx \, \phi_0 \mathcal{O} \rangle = Z_{\text{string}}[\phi(\partial \text{AdS}) = \phi_0]$$

The correspondence is simplest at strong coupling $g^2 N_c$

strongly coupled gauge theory ⇔ classical string theory
Holographic duals at finite temperature

Thermal (conformal) field theory $\equiv \text{AdS}_5$ black hole

CFT temperature $\Leftrightarrow$ Hawking temperature of black hole

CFT entropy $\Leftrightarrow$ Hawking-Bekenstein entropy

$s(\lambda \to \infty) = \frac{\pi^2}{2} N_c^2 T^3 = \frac{3}{4} s(\lambda = 0)$

Gubser and Klebanov
Holographic duals: Transport properties

Thermal (conformal) field theory \equiv AdS_5 black hole

\[ T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \quad g_{\mu\nu} = g_{\mu\nu}^0 + \gamma_{\mu\nu} \]

CFT entropy \Leftrightarrow \text{Hawking-Bekenstein entropy} \\
\sim \text{area of event horizon}

shear viscosity \Leftrightarrow \text{Graviton absorption cross section} \\
\sim \text{area of event horizon}
Holographic duals: Transport properties

Thermal (conformal) field theory $\equiv \text{AdS}_5$ black hole

CFT entropy $\iff$ Hawking-Bekenstein entropy

$\sim$ area of event horizon

Graviton absorption cross section

$\sim$ area of event horizon

Strong coupling limit

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}$$

Son and Starinets (2001)

Strong coupling limit universal? Provides lower bound for all theories?
Viscosity bound: Common fluids

\[ \frac{4\pi \eta}{\bar{h} s} \]

- Helium 0.1MPa
- Nitrogen 10MPa
- Water 100MPa

Viscosity bound

T, K
AdS/CFT low viscosity goo

pQCD kinetic plasma
Effective theories for fluids (Here: Weak coupling QCD)

\[ \mathcal{L} = \bar{q}_f (i \gamma^\mu D_\mu - m_f) q_f - \frac{1}{4} G^a_{\mu \nu} G^{a \mu \nu} \]

\[ \frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla} x f_p = C[f_p] \quad (\omega < T) \]

\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < g^4 T) \]
Effective theories (Strong coupling)

\[ \mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \ldots \Leftrightarrow S = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g} \mathcal{R} + \ldots \]

\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < T) \]
Kinetics vs No-Kinetics

Spectral function $\rho(\omega) = \text{Im}G_R(\omega, 0)$ associated with $T_{xy}$

\[ \frac{1}{s \rho^{xyy}(\omega)} \approx \frac{1}{g^4}, \quad \frac{1}{g^2}, \quad \frac{1}{\omega/T} \quad (\omega/T)^3 \]

weak coupling QCD

strong coupling AdS/CFT

transport peak vs no transport peak
Transport coefficients, theory

1. Kinetic theory
2. Kubo formula, lattice
3. Dynamic universality
4. Holography
Kinetic theory

Quasi-Particles ($\gamma \ll \omega$): introduce distribution function $f_p(x, t)$

$$N = \int d^3p f_p \quad T_{ij} = \int d^3p p_i p_j f_p,$$

Boltzmann equation

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p + \vec{F} \cdot \vec{\nabla}_p f_p = C[f_p]$$

Collision term $C[f_p] = C_{\text{gain}} - C_{\text{loss}}$

$$C_{\text{loss}} = \int dp' dq dq' f_p f_{p'} w(p, p'; q, q') \quad C_{\text{gain}} = \ldots$$
Linearized theory (Chapman-Enskog): \( f_p = f_p^0 (1 + \chi_p / T) \)

\[
RHS = C[f_p] \equiv \frac{f_p^0}{T} C_p \chi_p \quad \text{linear collision operator}
\]

Linear response to flow gradient

\[
f_p = \exp\left( - \left( E_p - \vec{p} \cdot \vec{v}(x) \right) / (kT) \right)
\]

Drift term proportional to “driving term” \( (v_{ij} = \partial_i v_j + \partial_j v_i - \text{trace}) \)

\[
LHS = \frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p \equiv \frac{f_p^0}{T} X \quad X \equiv p_i p_j v_{ij}
\]

Boltzmann equation

\[
C_p \chi_p = X \quad \chi_p \equiv g_p p_i p_j v_{ij} \equiv (\chi_p)_{ij} v_{ij}
\]
compute $T_{ij}[f_p^0 + \delta f_p] \equiv T_{ij}^0 + \eta v_{ij}$

$$\eta \sim \langle X|\chi\rangle \quad \langle X|\chi\rangle = \int d^3p f_p^0 (p_ip_j \chi^{ij}_p)$$

Use Boltzmann equation $C_p \chi_p = X$: \quad \eta \sim \langle \chi|C_p|\chi\rangle$

Variational principle

$$\langle \chi_{var}|C_p|\chi_{var}\rangle \langle \chi|C_p|\chi\rangle \geq \langle \chi_{var}|C_p|\chi\rangle^2 = \langle \chi_{var}|X\rangle^2$$

$$\eta \geq \frac{\langle \chi_{var}|X\rangle^2}{\langle \chi_{var}|C|\chi_{var}\rangle}$$

Best bound for $g_p \sim p^\alpha$ ($\alpha \simeq 0.1$)

$$\eta = \frac{0.34T^3}{\alpha_s^2 \log(1/\alpha_s)}$$

log($\alpha_s$) from dynamic screening

Baym et al. (1990)
pQCD and pSYM: weak versus strong coupling

Arnold, Dogan, Moore (2006)

Kinetic Theory: Quasiparticles

<table>
<thead>
<tr>
<th>Low Temperature</th>
<th>High Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>unitary gas</strong></td>
<td>phonons</td>
</tr>
<tr>
<td></td>
<td>atoms</td>
</tr>
<tr>
<td>helium</td>
<td>phonons, rotons</td>
</tr>
<tr>
<td></td>
<td>atoms</td>
</tr>
<tr>
<td>QCD</td>
<td>pions</td>
</tr>
<tr>
<td></td>
<td>quarks, gluons</td>
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</tbody>
</table>
Theory Summary

- **unitary gas**

\[ \eta/s \] vs. \[ T/T_F \]

- **\(^4\text{He}\)**

\[ \eta/s \] vs. \[ T[K] \]

- **QCD**

\[ \eta/s \] vs. \[ T[\text{MeV}] \]
What if the coupling is strong? Kubo Formula

Linear response theory provides relation between transport coefficients and Green functions

\[ G_R(\omega, 0) = \int dt \, d^3 x \, e^{i\omega t} \Theta(t) \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle \]

\[ \eta = - \lim_{\omega \to 0} \frac{1}{\omega} G_R(\omega, 0) \]

This result is hard to use for quantum fluids, but there are some heroic efforts by lattice QCD theorists, e.g. Meyer (2007).
\begin{align*}
\rho(\omega) K(x_0=1/2T,\omega)/T^4
\end{align*}

<table>
<thead>
<tr>
<th>$T$</th>
<th>$1.02$ $T_c$</th>
<th>$1.24$ $T_c$</th>
<th>$1.65$ $T_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta/s$</td>
<td>$0.102(56)$</td>
<td>$0.134(33)$</td>
<td></td>
</tr>
<tr>
<td>$\zeta/s$</td>
<td>$0.73(3)$</td>
<td>$0.065(17)$</td>
<td>$0.008(7)$</td>
</tr>
</tbody>
</table>
Dynamic Universality

Continuous phase transition: Dynamics of low energy modes universal

**Universality for transport coefficients**

Universal theory: Hydro (diffusive modes), order parameters (time dependent LG), stochastic forces (Langevin)

\[
\frac{\partial}{\partial t} (\rho v_i) = P_{ij}^\perp \left[ \eta_0 \nabla^2 \frac{\delta H}{\delta (\rho v_j)} + w_0 (\nabla_j \phi) \frac{\delta H}{\delta \phi} + \zeta_j \right]
\]

Model H of Hohenberg and Halperin

\[
\eta \sim \xi^{x_\eta} \quad (x_\eta \simeq 0.06) \quad \zeta \sim \xi^{x_\zeta} \quad (x_\zeta \simeq 2.8)
\]
Anti-DeSitter Space

Consider a hyperboloid embedded in 6-d euclidean space

\[-R^2 = \sum_{i=1,4} x_i^2 - x_0^2 - x_5^2\]

This is a space of constant negative curvature, and a solution of the Einstein equation

\[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}g_{\mu\nu}\Lambda\]

with negative cosmological constant.

Isometries of $AdS_5$ : $SO(4,2) \equiv$ conformal group in $d = 3 + 1$
metric of $\text{AdS}_5 \times S_5$ (note that $(L/\ell_s)^4 = g^2 N_c$)

\[ ds^2 = \frac{r^2}{L^2} (-dt^2 + dx^2) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2 \]

$r \to \infty$ “boundary” of $\text{AdS}_5$

Finite temperature: $\text{AdS}_5$ black hole solution

\[ ds^2 = \frac{r^2}{L^2} (-f(r)dt^2 + dx^2) + \frac{L^2}{f(r)r^2} dr^2 , \]

where $f(r) = 1 - (r_0/r)^4$. Hawking temperature $T_H = r_0/\pi$.

Compute induced stress tensor on the boundary

\[ \langle T_{\mu\nu} \rangle = \lim_{\epsilon \to 0} -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \bigg|_{bd} \]

Find ideal fluid with

\[ \langle T_{\mu\nu} \rangle = \text{diag}(\epsilon, P, P, P) , \quad \frac{\epsilon}{3} = P = \frac{N_c^2}{8\pi^2}(\pi T)^4 \]
Hydrodynamics from AdS/CFT

Eddington-Finkelstein coordinates

\[ ds^2 = 2 \, dv \, dr + \frac{r^2}{L^2} \left[ -f(r) \, dv^2 + r^2 \, dx^2 \right] \]

Introduce local rest frame \( u^\mu = (1, 0) \), scale parameter \( b \)

\[ ds^2 = -2u_\mu dx^\mu \, dr + \frac{r^2}{L^2} \left[ -f(br)u_\mu u_\nu \, dx^\mu \, dx^\nu + r^2 P_{\mu \nu} \, dx^\mu \, dx^\nu \right] \]

\[ P_{\mu \nu} = u_\mu u_\nu + \eta_{\mu \nu} \]

promote \( u_\mu(x) \) and \( b(x) \) to fields
determine metric order by order in gradients
compute induced stress
leading order: ideal fluid dynamics with $\epsilon = 3P$

$$T_{0}^{\mu\nu} = \frac{N_{c}^{2}}{8\pi^{2}} (\pi T)^{4} (\eta^{\mu\nu} + 4u^{\mu}u^{\nu})$$

next-to-leading order: Navier-Stokes with $\eta/s = 1/(4\pi)$

$$\delta^{(1)} T^{\mu\nu} = -\frac{N_{c}^{2}}{8\pi^{2}} (\pi T)^{3} \sigma^{\mu\nu}$$

next-to-next-to-leading order: second order conformal hydro

$$\delta^{(2)} T^{\mu\nu} = \eta \tau_{II} \left[ \langle D\sigma^{\mu\nu} \rangle + \frac{1}{3} \sigma^{\mu\nu} (\partial \cdot u) \right]$$

$$+ \lambda_{1} \sigma^{\langle \mu \sigma^{\nu} \rangle}_{\lambda} + \lambda_{2} \sigma^{\langle \mu \Omega^{\nu} \rangle}_{\lambda} + \lambda_{3} \Omega^{\langle \mu \Omega^{\nu} \rangle}_{\lambda}$$

relaxation times

$$\tau_{II} = \frac{2 - \ln 2}{\pi T} \quad \lambda_{1} = \frac{2\eta}{\pi T} \quad \lambda_{2} = \frac{2\eta \ln 2}{\pi T} \quad \lambda_{3} = 0$$