Nearly Perfect Fluidity:
From Atoms to Quarks

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Hydrodynamics: Long-wavelength, low-frequency dynamics of conserved or spontaneously broken symmetry variables.

Historically: Water

\[(\rho, \epsilon, \vec{\pi})\]
Simple non-relativistic fluid

Simple fluid: Conservation laws for mass, energy, momentum

\[
\frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) = 0
\]

\[
\frac{\partial \epsilon}{\partial t} + \nabla j^\epsilon = 0
\]

\[
\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0
\]

Constitutive relations: Energy momentum tensor

\[
\Pi_{ij} = P \delta_{ij} + \rho v_i v_j + \eta \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2)
\]

reactive  dissipative  2nd order

Expansion \( \Pi_{ij}^0 \gg \delta \Pi_{ij}^1 \gg \delta \Pi_{ij}^2 \)
Regime of applicability

Expansion parameter \( Re^{-1} = \frac{\eta(\partial v)}{\rho v^2} = \frac{\eta}{\rho Lv} \ll 1 \)

\[
Re = \frac{\hbar n}{\eta} \times \frac{mvL}{\hbar}
\]

fluid property \quad flow property

Kinetic theory estimate: \( \eta \sim npl_{mfp} \)

\[
Re^{-1} = \frac{v}{c_s} Kn \\
Kn = \frac{l_{mfp}}{L}
\]

expansion parameter \( Kn \ll 1 \)
Shear viscosity

Viscosity determines shear stress ("friction") in fluid flow

$$F = A \eta \frac{\partial v_x}{\partial y}$$

Kinetic theory: conserved quantities carried by quasi-particles

$$\frac{\partial f_p}{\partial t} + \bar{v} \cdot \nabla_x f_p + \bar{F} \cdot \nabla_p f_p = C[f_p]$$

$$\eta \sim \frac{1}{3} n \bar{p} l_{mfp}$$

Dilute, weakly interacting gas: $l_{mfp} \sim 1/(n\sigma)$

$$\eta \sim \frac{1}{3} \frac{\bar{p}}{\sigma}$$

independent of density!
Shear viscosity

non-interacting gas ($\sigma \rightarrow 0$): $\eta \rightarrow \infty$

non-interacting and hydro limit ($T \rightarrow \infty$) limit do not commute

strongly interacting gas:

$$\frac{\eta}{n} \sim \bar{p}l_{mfp} \geq \hbar$$

but: kinetic theory not reliable!

what happens if the gas condenses into a liquid?

Eyring, Frenkel:

$$\eta \simeq hn \exp\left(\frac{E}{T}\right) \geq hn$$
Holographic Duals: Transport Properties

Thermal (conformal) field theory $\equiv AdS_5$ black hole

CFT temperature $\iff$ Hawking temperature

CFT entropy $\iff$ Hawking-Bekenstein entropy

$\sim$ area of event horizon

shear viscosity $\iff$ Graviton absorption cross section

$\sim$ area of event horizon

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

$$g_{\mu\nu} = g^{0}_{\mu\nu} + \gamma_{\mu\nu}$$
Holographic Duals: Transport Properties

Thermal (conformal) field theory \( \equiv \text{AdS}_5 \) black hole

CFT entropy \( \iff \) Hawking-Bekenstein entropy
\( \sim \) area of event horizon

shear viscosity \( \iff \) Graviton absorption cross section
\( \sim \) area of event horizon

Strong coupling limit

\[
\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}
\]

Son and Starinets (2001)

Strong coupling limit universal? Provides lower bound for all theories?
Kinetics vs No-Kinetics

AdS/CFT low viscosity goo

pQCD kinetic plasma
Effective theories for fluids (Here: Weak coupling QCD)

\[ \mathcal{L} = \bar{q}_f (i \slashed{D} - m_f) q_f - \frac{1}{4} G^a_{\mu \nu} G^{a}_{\mu \nu} \]

\[ \frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \quad (\omega < T) \]

\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < g^4 T) \]
Effective theories (Strong coupling)

\[ \mathcal{L} = \bar{\lambda}(i \sigma \cdot D) \lambda - \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + \ldots \iff S = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g} \mathcal{R} + \ldots \]

\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < T) \]
Kinetics vs No-Kinetics

Spectral function $\rho(\omega) = \text{Im} G_R(\omega, 0)$ associated with $T_{xy}$

weak coupling QCD

strong coupling AdS/CFT

transport peak vs no transport peak
Spectral function (lattice QCD)

\[ \rho(\omega) K(x_0=1/2T, \omega)/T^4 \]

\begin{tabular}{|c|c|c|c|}
\hline
\textbf{T} & 1.02 \text{T}_c & 1.24 \text{T}_c & 1.65 \text{T}_c \\
\hline
\text{\eta/s} & 0.102(56) & 0.134(33) &
\hline
\text{\zeta/s} & 0.73(3) & 0.065(17) & 0.008(7) \\
\hline
\end{tabular}

H. Meyer (2007)
Perfect Fluids: How to be a contender?

Bound is quantum mechanical

need quantum fluids

Bound is incompatible with weak coupling and kinetic theory

strong interactions, no quasi-particles

Model system has conformal invariance (essential?)

(Almost) scale invariant systems
Perfect Fluids: The contenders

QGP (T=180 MeV)

Trapped Atoms
(T=0.1 neV)

Liquid Helium
(T=0.1 meV)
Perfect Fluids: The contenders

QGP $\eta = 5 \cdot 10^{11} \text{Pa} \cdot \text{s}$

Trapped Atoms
$\eta = 1.7 \cdot 10^{-15} \text{Pa} \cdot \text{s}$

Liquid Helium
$\eta = 1.7 \cdot 10^{-6} \text{Pa} \cdot \text{s}$

Consider ratios $\eta/s$
# Kinetic Theory: Quasiparticles

<table>
<thead>
<tr>
<th>Low Temperature</th>
<th>High Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Unitary Gas</strong></td>
<td>phonons</td>
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<tr>
<td><strong>Helium</strong></td>
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<td>pions</td>
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<td><img src="image5" alt="Diagram" /></td>
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</tbody>
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Theory Summary

- Unitary gas
- $^4$He
- QCD

Graphs showing the behavior of $\eta/s$ as a function of $T/T_F$, $T[K]$, and $T[MeV]$. The graphs illustrate the variation of $\eta/s$ with temperature for different systems.
I. Experiment (Liquid Helium)

Kapitza (1938)
viscosity vanishes below $T_c$
capillary flow viscometer

Hollis-Hallett (1955)
roton minimum, phonon rise
rotation viscometer

$\eta/s \simeq 0.8 \hbar/k_B$
II. Hydrodynamics (Cold atoms)

Radial breathing mode

Ideal fluid hydrodynamics \( (P \sim n^{5/3}) \)

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \vec{v}) = 0
\]

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla P}{mn} - \frac{\nabla V}{m}
\]

Hydro frequency at unitarity

\[
\omega = \sqrt{\frac{10}{3}} \omega_\perp
\]

Damping small, depends on \( T/T_F \).

experiment: Kinast et al. (2005)
Viscous Hydrodynamics

Energy dissipation ($\eta, \zeta, \kappa$: shear, bulk viscosity, heat conductivity)

$$\dot{E} = -\frac{1}{2} \int d^3x \eta(x) \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right)^2$$

$$- \int d^3x \zeta(x) (\partial_i v_i)^2 - \frac{1}{T} \int d^3x \kappa(x) (\partial_i T)^2$$

Shear viscosity to entropy ratio
(assuming $\zeta = \kappa = 0$)

$$\eta_s = \left(3\lambda N\right)^{1/3} \frac{\Gamma E_0}{\omega_\perp E_F N S}$$

Schaefer (2007), see also Bruun, Smith

$T \ll T_F \quad T \gg T_F, \tau_R \simeq \eta/P$
Dissipation

$R_i$ [µm] vs $t$[ms]

$R_\perp$, $R_z$

$\theta$ [°]

$E/E_F = 0.56$

$E/E_F = 2.1$

$R_\perp$

$R_\perp$, $R_z$

Dissipation

\[
\frac{\delta t_0}{t_0} = \left\{ \begin{array}{c} 0.008 \\ 0.024 \end{array} \right\} \left( \frac{\langle \alpha_s \rangle}{1/(4\pi)} \right) \left( \frac{2 \cdot 10^5}{N} \right)^{1/3} \left( \frac{S/N}{2.3} \right) \left( \frac{0.85}{E_0/E_F} \right)
\]

\( t_0 \): “Crossing time” \( b_\perp = b_z, \theta = 45^\circ \)
\( a \): amplitude
Time Scales

$t_{acc}$ | $t_{diss}$ | $t_{fr}$ | $t_{cross}$

$R_i [\mu m]$ | $R_\perp$ | $R_z$

$t [ms]$
Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy.

\[ p_0 \left. \frac{dN}{d^3p} \right|_{p_z=0} = v_0(p_\perp) \left( 1 + 2v_2(p_\perp) \cos(2\phi) + \ldots \right) \]
Elliptic flow: initial entropy scaling

Viscosity and Elliptic Flow

Consistency condition $T_{\mu\nu} \gg \delta T_{\mu\nu}$
(applicability of Navier-Stokes)

$\eta + \frac{4}{3} \zeta \ll \frac{3}{4} (\tau T)$

Danielewicz, Gyulassy (1985)

Very restrictive for $\tau < 1$ fm

Many questions: Dependence on initial conditions, freeze out, etc.

conservative bound $\frac{\eta}{s} < 0.4$
Remarkably, the best fluids that have been observed are the coldest and the hottest fluid ever created in the laboratory, cold atomic gases \( (10^{-6} \text{K}) \) and the quark gluon plasma \( (10^{12} \text{K}) \) at RHIC.

Both of these fluids come close to a bound on the shear viscosity that was first proposed based on calculations in string theory, involving non-equilibrium evolution of back holes in 5 (and more) dimensions.