Instantons, Large $\mathcal{N}_c$, and Holographic Models of QCD

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Outline

• Introduction: Why Instantons?
  $U(1)_A$ puzzle, topology, and instantons

• Instantons and the large $N_c$ limit
  smooth large $N_c$ limit? Witten-Veneziano relation?

• $U(1)_A$ problem in a holographic model of QCD
  Relation to Instantons?
**U(1)_A Puzzle**

QCD has a $U(1)_A$ symmetry $\psi_L \rightarrow e^{i\phi}\psi_L$, $\psi_R \rightarrow e^{-i\phi}\psi_R$

spontaneously broken $\langle \bar{\psi}_L \psi_R \rangle = \Sigma \neq 0$

Goldstone Theorem: massless Goldstone boson $m_{\eta'} \rightarrow 0$ ($m_q \rightarrow 0$)

But: $\eta'$ is heavy, $m_{\eta'}^2 \gg m_q\Lambda_{QCD}$

Resolution: axial anomaly

$$\partial_\mu A^\mu = \frac{N_f}{16\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

But: RHS is a total divergence $G\tilde{G} \sim \partial^\mu K_\mu$

$\Rightarrow$ Topology is important
Topology in QCD

classical potential is periodic in variable $X$

$$X = \int d^3x \, K_0(x, t)$$

$$\partial^\mu K_\mu = \frac{1}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$
classical minima correspond to pure gauge configurations

\[ A_i(x) = iU^\dagger(x)\partial_i U(x) \]

\[ E^2 = B^2 = 0 \]

classical tunneling paths: Instantons

\[ A^a_\mu(x) = 2\frac{\eta^a_{\mu\nu}x_\nu}{x^2 + \rho^2}, \]

\[ G^a_\mu \tilde{G}^a_\mu = \frac{192\rho^4}{(x^2 + \rho^2)^4}. \]
Dirac spectrum in the field of an instanton

\[ \Delta Q_A = 2 \]

axial charge violation:

Instanton induced quark interaction \((N_f = 2)\)

\[ \mathcal{L} = G \det_f (\bar{\psi}_L, f \psi_R, g) \]

violates \(U(1)_A\) but preserves \(SU(2)_{L,R}\)

\[ G = \int d\rho \, n(\rho) \]

\(\ldots\) and contributes to the \(\eta'\) mass
Tunneling rate (barrier penetration factor)

\[ n(\rho) \sim \exp \left[ -\frac{8\pi^2}{g^2(\rho)} \right] \sim \rho^{b-5} \]
QCD at large $N_c$

QCD ($m = 0$) is a parameter free theory. Very beautiful.

But: No expansion parameter

’t Hooft: Consider $N_c \to \infty$ and use $1/N_c$ as a small parameter

$N_c \to \infty \quad \Rightarrow \quad$ classical master field

keep $\Lambda_{QCD}$ fixed $\Rightarrow g^2 N_c = const$

Could the master field be a multi-instanton?

Witten: No! $\quad dn \sim \exp \left(-\frac{1}{g^2}\right) \sim \exp (-N_c)$
\( U(1)_A \) anomaly at large \( N_c \)

consider \( \theta \) term

\[
\mathcal{L} = \frac{ig^2\theta}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}
\]

no \( \theta \) dependence in perturbation theory.

\textbf{Witten: non-perturbative \( \theta \) dependence}

\[
\chi_{\text{top}} = \frac{d^2E}{d\theta^2} \bigg|_{\theta=0} \sim O(1)
\]

massless quarks: topological charge screening

\[
\lim_{m \to 0} \chi_{\text{top}} = 0
\]

How can that happen? Fermion loops are suppressed!

Witten:

\(
\eta' \text{ has to become light}
\)

\[
f_\pi m_{\eta'}^2 = 2N_f \chi_{\text{top}}(\text{no quarks})
\]

\[
\Rightarrow \quad m_{\eta'}^2 = O(1/N_c)
\]
First Part

The instanton ensemble at large $N_c$
**Instantons at large $N_c$**

semi-classical ensemble of instantons at large $N_c$

![Diagram of instantons in color and coordinate space]

instantons are $N_c = 2$ configurations

\[
\left( \frac{N}{V} \right) = O(N_c)
\]

\[\Rightarrow \epsilon_{vac} = O(bN_c) = O(N_c^2)\]

instantons are semi-classical

\[\rho \simeq \rho^* = O(1) \quad S_{inst} = O(N_c)\]

density $dn \sim \exp(-S_{inst}) = O(\exp(-N_c))$?

*NO! large entropy* $dn \sim \exp(+N_c)$

topological susceptibility $\chi_{top} \simeq (N/V) = O(N_c)$?

*NO! fluctuations suppressed* $\chi_{top} = O(1)$
Instanton ensemble

\[
Z = \frac{1}{N_I!N_A!} \prod_{I} \int [d\Omega_I n(\rho_I)] e^{-S_{int}}
\]

complicated \(N_c\) dependence

\[
n(\rho) = C_{N_c} \left( \frac{8\pi^2}{g^2} \right)^{2N_c} \rho^{-5} \exp \left[ -\frac{8\pi^2}{g(\rho)^2} \right]
\]

\[
C_{N_c} = \frac{0.47 \exp(-1.68N_c)}{(N_c-1)!(N_c-2)!}
\]
\[
\frac{8\pi^2}{g^2(\rho)} = -b \log(\rho\Lambda), \quad b = \frac{11}{3} N_c
\]

\[
S_{int} = -\frac{32\pi^2}{g^2} |u|^2 \left\{ \frac{\rho_I^2 \rho_A^2}{R_{IA}^4} (1 - 4 \cos^2 \theta) + S_{core} \right\}
\]
complicated ensemble, size distribution

\[ n(\rho) \sim \begin{cases} \exp(-N_c) & \rho < \rho^* \\ \text{const} & \rho \sim \rho^* \end{cases} \quad \rho^* \sim O(1) \]

total density determined by interactions

\[ S(1 - \text{body}) \sim N_c \sim N_c \times \frac{1}{N_c} \times \left( \frac{N}{V} \right) \sim \text{classical} \times \text{color overlap} \times \text{density} \]

conclude

\[ \left( \frac{N}{V} \right) = O(N_c) \]
fluctuations in $N$ are $1/N_c$ suppressed

$$\langle N^2 \rangle - \langle N \rangle^2 = \frac{4}{b} \langle N \rangle \sim O(1) \quad \text{(not } O(N_c)\text{!)}$$

also true for topological charge

$$\langle Q^2 \rangle = \frac{4}{b-\pi(b-4)} \langle N \rangle \sim O(1)$$
global observables: $(N/V), \langle \bar{q}q \rangle, \chi_{top}$
Second Part

Holographic QCD
QCD and Strings: Pre-History

\[ J \sim \alpha' M^2 \]

\[ T_{str} = \frac{1}{2\pi\alpha'} = 1 \text{ GeV/fm} \]

Regge trajectories, pomerons, dual models, Veneziano amplitude, \ldots

faded away with the advent of QCD
QCD and Strings: Pre-History

’t Hooft: large $N_c$ expansion ($\lambda = g^2 N_c = \text{const}$)

\[
\sim (g_{YM}^2)^{8-2} N_c^3 = \lambda N_c^2
\]

\[
\sim (g_{YM}^2)^{6-4} N_c^4 = \lambda^2 N_c^2
\]

\[
\sim (g_{YM}^2)^{8-5} N_c^5 = \lambda^3 N_c^2
\]

\[
\sim (g_{YM}^2)^{6-4} N_c^2 = \lambda^2
\]

Large $N_c$ limit: topological expansion (string theory?)
QCD and Strings: Pre-History

QCD: flux tubes and string potentials
The AdS/CFT duality relates

\[ \mathcal{N} = 4 \text{ large } N_c \text{ gauge theory in 4 dimensions} \]

\[ \iff \]

type IIb string theory on \( AdS_5 \times S_5 \)

\[ \iff \]

boundary correlation fcts of AdS fields

\[ \langle \exp \int dx \phi_0 \mathcal{O} \rangle = Z_{string}[\phi(\partial AdS) = \phi_0] \]

The correspondence is simplest at strong coupling \( g^2 N_c \)

strongly coupled gauge theory \( \iff \) classical string theory

Maldacena (1997)
\( \mathcal{N} = 4 \) Supersymmetric Yang-Mills Theory

Fields: Gluons, Gluinos, Higgses; all in the adjoint of \( SU(N_c) \)

\[
\mathcal{L} = \frac{1}{4} (F_{\mu\nu}^a)^2 + \bar{\lambda}^a_A \sigma^\mu (D_\mu \lambda^A)^a + (D_\mu \Phi_{AB})^a (D_\mu \Phi^{AB})^a + \ldots
\]

\[
A^a_\mu \quad \lambda^a_A \quad (\bar{4}_R) \quad \Phi^a_{AB} \quad (6_R)
\]

Global symmetries: Conformal and \( SU(4)_R \)

\( SO(4,2) \times SU(4)_R \)

Properties: Conformal \( \beta(g) = 0 \), extra scalars, no fundamental fermions, no chiral symmetry breaking, no confinement
QCD and Strings: Towards QCD

non-AdS/non-CFT correspondence, a.k.a “AdS/QCD”

AdS: conformal  
cutoff AdS  
AdS black hole
Consider five dimensional action \((F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu])\)

\[
S_5 = -\frac{1}{4g_5^2} \int d^5x \sqrt{g} F^a_{\mu\nu} F^{a\mu\nu}
\]

\[
ds^2 = \frac{1}{z^2}(-dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu)
\]

\(0 \leq z \leq z_m\)

Equ. of motion: linearize, F-trafo in \(x^\mu\), \(V_5 = 0\) gauge

\[
z \partial_z \left( \frac{1}{z} \partial_z V^a_\mu \right) + q^2 V^a_\mu = 0
\]

Using equ. of motion

\[
S_5 = -\frac{1}{2g_5^2} \int d^4x \frac{1}{z} V^a_\mu \partial_z V^a_\mu \bigg|_{z=0}
\]
1. Find solution with $V(z \to 0, x) = V_0(x)$
2. Compute action $S_5[V]$.
3. Take functional derivative $\Pi_{\mu\nu} = (\delta^2 S_5)/(\delta V_0^\mu \delta V_0^\nu)$

Write $V^\mu(q, z) = V_0^\mu(q) V(q, z)$ with $V(q, 0) = 1$. Then

$$\Pi(Q^2) = -\frac{1}{g_5^2 Q^2} \frac{\partial_z V(q, z)}{z} \bigg|_{z=0}$$

$$Q^2 = -q^2$$

The required solution is

$$V(q, z) \simeq 1 + \frac{1}{2} Q^2 z^2 \log(Qz) + \ldots$$

$$\Pi(Q^2) = -\frac{1}{2 g_5^2} \log(Q^2)$$

match to QCD:

$$g_5^2 = \frac{12\pi^2}{N_c}$$
\textbf{AdS/CFT Dictionary}

4d field theory $\leftrightarrow$ 5d gravitational theory

generating functional $W[\phi_0] \leftrightarrow$ boundary action $S[\phi_0]$

operator $O(x)$ coupled to $\phi_0(x) \leftrightarrow$ field $\phi(z, x)$ (boundary val $\phi_0(x)$)

dimension, spin of $O \leftrightarrow$ 5-d mass of $\phi$

symmetry breaking: $\leftrightarrow$ non-normalizable mode:

$\langle O \rangle \neq 0$ as $\phi_0 \rightarrow 0$  $\phi \sim \phi_0 z^d \phi + A z^d O$

large $N_c \leftrightarrow$ weak coupling $g_5$

large $Q \leftrightarrow$ small $z$
The model: Chiral Symmetry Breaking

5-d action with vector and scalar fields

\[ S = \int d^5 x \sqrt{g} \left\{ -\frac{1}{4g_5^2} \text{Tr} \left( F_L^2 + F_R^2 \right) + \text{Tr} \left( |DX|^2 + 3|X|^2 \right) \right\} \]

Erlich et al. (2005), DaRold et al (2005)

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \text{ (for L/R)} \]

\[ X \rightarrow LXR \text{ and } D_\mu X = \partial_\mu X - iA_{L\mu}X + iX A_{R\mu} \]

Chiral symmetry breaking

\[ \langle X_{ij} \rangle = \sigma_{ij} z^3 + M_{ij} z, \]

Pseudoscalar fields

\[ X_{ij} = \langle X_{ij} \rangle \exp(i\pi^a t^a), \]
Chiral Symmetry Breaking and the Pion

Mixing between axial and pseudoscalars: \( A_\mu = A_\mu \perp + \partial_\mu \varphi \)

\[
\partial_z \left( \frac{1}{z} \partial_z A_\perp^a \right) + \frac{q^2}{z} A_\perp^a - \frac{g_5^2 v^2}{z^3} A_\perp^a = 0
\]

\[
\partial_z \left( \frac{1}{z} \partial_z \varphi^a \right) + \frac{g_5^2 v^2}{z^3} (\pi^a - \varphi^a) = 0. \quad [v(z) = mz + \sigma z^3]
\]

\[-q^2 \partial_z \varphi^a + \frac{g_5^2 v^2}{z^2} \partial_z \pi^a = 0. \]

Goldstone mode: Define \( f_\pi^2 = - \frac{1}{g_5^2} \frac{\partial_z A(z,0)}{z} \bigg|_{z=0} \) (b.c. \( A(0, q) = 1 \))

\[
\phi(z) = A(0, z) - 1 \quad (\pi(z) = 1)
\]

\[
\pi(z) = q^2 \int_0^z \frac{d\bar{z}}{v(\bar{z})} \frac{\bar{z}^3}{g_5^2 \bar{z}} \partial_{\bar{z}} A(0, \bar{z})
\]

\[
m_\pi^2 f_\pi^2 = 2m\sigma
\]
Vector/Axialvector Correlation Functions

Data: V/A spectral functions from $\tau \rightarrow \nu_\tau + \text{hadrons}$ (Aleph)
Flavor Singlet Axialvector

Add singlet field $Y = \langle Y \rangle e^{i\alpha}$ (pseudoscalar glueball, “axion”)

$$S = \int d^5x \sqrt{g} \left\{ \frac{1}{2}|DY|^2 + \frac{\kappa_0}{2} (Y^{N_f} \det(X) + h.c.) \right\}$$

Katz & Schwartz (2007)

$$Y = \langle Y \rangle = c + \Xi z^4 \quad c \sim g^2, \quad \Xi \sim G^2$$

QCD axial anomaly:

$$\partial^\mu j_5^\mu = 2N_f \frac{g^2}{32\pi^2} G\tilde{G}$$

$$\int d^4x \ e^{iqx} \langle \partial^\mu j_5^\mu(x)\partial^\mu j_5^\mu(0) \rangle = (2N_f)^2 \frac{\alpha_s^2}{8\pi^4} Q^4 \log(Q^2) + \ldots$$

Matching: axial fields $A_\mu^0 = A_{\mu \perp}^0 + \partial_\mu \varphi^0$ and $a$

$$\int d^4x \ e^{iqx} \langle \partial^\mu j_5^\mu(x)\partial^\mu j_5^\mu(0) \rangle = -\frac{Q^2}{g_5^2} \frac{\partial_z \varphi^0(z)}{z} \bigg|_{z=0}$$

$$c = \sqrt{2N_f} \frac{\alpha_s}{2\pi^2} \quad \Xi \rightarrow \text{OPE} \quad \kappa \text{ free}$$
Spectrum: Pseudoscalar Singlets

eigenvalues of \( (\varphi^0, \eta^0, a) \) and \( (\varphi, \pi) \) system

---

excited state (mostly \( \bar{q}q \))

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excited state (mostly \( G\tilde{G} \))

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Pseudoscalar Correlation Functions

\[ \Pi(x) = \langle \bar{q} t^a \gamma_5 q(x) \bar{q} t^b \gamma_5 q(0) \rangle \]

\[ \Pi(x) = \left\langle g^2 G \tilde{G}(x) g^2 G \tilde{G}(0) \right\rangle \]

\[ m_{\eta'} \simeq 660 \text{ MeV} \]

\[ m_{0^+-} \simeq 1400 \text{ MeV} \]

\[ \langle 0 | g^2 G \tilde{G} | \eta' \rangle \neq 0 \]
Topological Susceptibility

Topological susceptibility

\[
\chi_{\text{top}} = \lim_{V \to \infty} \frac{\langle Q_{\text{top}}^2 \rangle}{V} = - \int d^4 x \, \Pi_P(x)
\]

Holography: Find solution with \(q^2 = 0\) and \(a(0, q) = 1\)

\[
\chi_{\text{top}} = - \frac{c^2}{2N_f} \frac{\partial_z a}{z^3} \bigg|_{\epsilon}
\]

\[
\partial_z \left( \frac{c^2}{z^3} \partial_z a \right) + \kappa \frac{v^{N_f}}{z^5} (\eta^0 - a) = 0
\]

\[
v^2 \partial_z \eta^0 + c^2 \partial_z a = 0
\]

Note that \(\chi_{\text{top}} \sim m_q \sigma\).
Witten-Veneziano

Pure gluodynamics

\[ a(z) = \frac{N_f}{2c^2} \chi_{top} z^4 + \ldots \]

Full QCD: (Pseudo) Goldstone modes \( \eta - \varphi \)

\[ \eta^0(z) \simeq 1 \quad \varphi^0(z) \simeq \frac{g_5^2}{2} f_\pi^2 z^2 + \ldots \]

Study coupling, use perturbation theory in \( c (\sim 1/N_c) \)

\[ m_{\eta'}^2 z^2 \partial_z \varphi^0 - g_5^2 v^2 \partial_z \eta^0 - g_5^2 c^2 \partial_z a = 0 \]

Witten-Veneziano relation

\[ f_{\eta'}^2 m_{\eta'}^2 = 4N_f \chi_{top} \]
What about instantons?

Topological charge correlator: Trear $\kappa a^2$ as a perturbation

$$\Pi_P(Q) = -\frac{1}{2N_f} \int_0^{\tilde{z}^m} \frac{dz}{z^5} \tilde{\kappa} \left[ \frac{1}{2} (Qz)^2 K_2(Qz) \right]^2,$$

$AdS_5$ measure $\times (\text{Bulk-to-boundary prop})^2$

Compare to instanton result

$$\Pi_P(Q) = -2 \int \frac{d\rho}{\rho^5} d(\rho) \left[ \frac{1}{2} (Q\rho)^2 K_2(\rho Q) \right]^2,$$

instanton measure $\times (\text{F-trafo of } G\tilde{G}_I)^2$

- AdS cutoff provides instanton size cutoff
- Correspondence extends to other correlators
Positivity and all that

\[ \chi_{top} = \lim_{V \to \infty} \frac{\langle Q_{top}^2 \rangle}{V} = - \int d^4x \, \Pi_P(x) \]

Have \( \chi_{top} \geq 0 \) and \( \Pi_P(x) \geq 0 \) (spectral positivity)

How can that be? \( \Pi_P(x) \sim \alpha_s^2/x^8 \) singular \( \Rightarrow \) need regulator

\[ \Pi_P^{reg}(x) = \Pi_P^{AdS|}(x) - \Pi_P^{AdS}(x) \]

\[ \int d^4x \, \Pi_P^{reg}(x) = - \frac{c^2}{2N_f} \left. \frac{\partial z \alpha}{z^3} \right|_\epsilon \]

Anomaly term: \( \delta \Pi_P(x) \leq 0 \) (\( \chi_{top} \geq 0 \))
Outlook

Improved models: Asymptotic freedom? OPE?


Top-down approach: Origin of anomaly term?


Large $N_c$ limit: Lattice/instanton calculations suggest that $d(\rho) \rightarrow \delta(\rho - \rho^*)$.

Teper (2003), Schäfer (2003), Shuryak (2007)

Non-zero $T, \mu$: Better perturbative control.
Holographic duals?