In Search of the Perfect Fluid

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Measures of Perfection

Viscosity determines shear stress ("friction") in fluid flow

$$ F = A \eta \frac{\partial v_x}{\partial y} $$

Dimensionless measure of shear stress: Reynolds number

$$ Re = \frac{n}{\eta} \times mvr $$

- $[\eta/n] = \hbar$
- Relativistic systems $Re = \frac{s}{\eta} \times \tau T$
Other sources of dissipation (thermal conductivity, bulk viscosity, ...) vanish for certain fluids, but shear viscosity is always non-zero.

There are reasons to believe that $\eta$ is bounded from below by a constant times $\hbar s/k_B$. In a large class of theories $\eta/s \geq \hbar/(4\pi k_B)$.

A fluid that saturates the bound is a “perfect fluid”.
Fluids: Gases, Liquids, Plasmas, …

Hydrodynamics: Long-wavelength, low-frequency dynamics of conserved or spontaneously broken symmetry variables.

\[ \tau \sim \tau_{micro} \]

\[ \tau \sim \lambda^{-1} \]

Historically: Water 
\[ (\rho, \epsilon, \vec{\pi}) \]
Example: Simple Fluid

Conservation laws: mass, energy, momentum

\[
\frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) = 0
\]

\[
\frac{\partial \epsilon}{\partial t} + \nabla \vec{j}^{\epsilon} = 0
\]

\[
\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0
\]

[Euler/Navier-Stokes equation]

Constitutive relations: Energy momentum tensor

\[
\Pi_{ij} = P \delta_{ij} + \rho v_i v_j + \eta \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2)
\]

reactive  dissipation  2nd order
Kinetic Theory

Kinetic theory: conserved quantities carried by quasi-particles

\[
\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p + \vec{F} \cdot \vec{\nabla}_p f_p = C[f_p]
\]

\[
\eta \sim \frac{1}{3} n \bar{p} l_{mfp}
\]

Normalize to density. Uncertainty relation suggests

\[
\frac{\eta}{n} \sim \bar{p} l_{mfp} \geq \hbar
\]

Also:

\[
s \sim k_B n \quad \text{and} \quad \eta/s \geq \hbar/k_B
\]

Validity of kinetic theory as \(\bar{p} l_{mfp} \sim \hbar\)?
Effective Theories for Fluids (Here: Weak Coupling QCD)

\[ \mathcal{L} = \bar{q}_f (iD - m_f) q_f - \frac{1}{4} G^\alpha_{\mu\nu} G^{\alpha}_{\mu\nu} \]

\[ \frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \quad (\omega < T) \]

\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < g^4 T) \]
Holographic Duals: Transport Properties

Thermal (conformal) field theory $\equiv AdS_5$ black hole

CFT entropy $\Leftrightarrow$ Hawking-Bekenstein entropy

$\sim$ area of event horizon

shear viscosity $\Leftrightarrow$ Graviton absorption cross section

$\sim$ area of event horizon

\[
T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}}
\]

\[
g_{\mu\nu} = g_{\mu\nu}^0 + \gamma_{\mu\nu}
\]
Holographic Duals: Transport Properties

Thermal (conformal) field theory \(\equiv AdS_5\) black hole

\[\begin{align*}
\text{CFT entropy} & \iff \text{Hawking-Bekenstein entropy} \\
\sim \text{area of event horizon} & \iff \text{Graviton absorption cross section} \\
\sim \text{area of event horizon}
\end{align*}\]

Strong coupling limit

\[\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}\]

Son and Starinets (2001)

Strong coupling limit universal? Provides lower bound for all theories?
Effective Theories (Strong coupling)

\[ \mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \ldots \Leftrightarrow S = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g} \mathcal{R} + \ldots \]

\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < T) \]
**Kinetics vs No-Kinetics**

- **AdS/CFT low viscosity goo**
- **pQCD kinetic plasma**
Kinetics vs No-Kinetics

Spectral function $\rho(\omega) = \text{Im} G_R(\omega, 0)$ associated with $T_{xy}$

weak coupling QCD

strong coupling AdS/CFT

transport peak vs no transport peak
Perfect Fluids: How to be a contender?

Bound is quantum mechanical

need quantum fluids

Bound is incompatible with weak coupling and kinetic theory

strong interactions, no quasi-particles

Model system has conformal invariance (essential?)

(Almost) scale invariant systems
Perfect Fluids: The contenders

QGP (T=180 MeV)

Trapped Atoms
(T=0.1 neV)

Liquid Helium
(T=0.1 meV)
Perfect Fluids: The contenders

QGP $\eta = 5 \cdot 10^{11} \text{Pa} \cdot \text{s}$

Trapped Atoms
$\eta = 1.7 \cdot 10^{-15} \text{Pa} \cdot \text{s}$

Liquid Helium
$\eta = 1.7 \cdot 10^{-6} \text{Pa} \cdot \text{s}$

Consider ratios $\eta/s$
## Kinetic Theory: Quasiparticles

<table>
<thead>
<tr>
<th></th>
<th>low temperature</th>
<th>high temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>unitary gas</td>
<td>phonons</td>
<td>atoms</td>
</tr>
<tr>
<td>helium</td>
<td>phonons, rotons</td>
<td>atoms</td>
</tr>
<tr>
<td>QCD</td>
<td>pions</td>
<td>quarks, gluons</td>
</tr>
</tbody>
</table>
**Low T: Phonons** Goldstone boson $\psi \psi = e^{2i \varphi \langle \psi \psi \rangle}$

$$\mathcal{L} = c_0 m^{3/2} \left( \mu - \dot{\varphi} - \frac{\langle \vec{\nabla} \varphi \rangle^2}{2m} \right)^{5/2} + \ldots$$

Viscosity dominated by $\varphi + \varphi \to \varphi + \varphi$

$$\eta = A \frac{\xi^5}{c_s^3} \frac{T_F^8}{T^5}$$


**High T: Atoms** Cross section regularized by thermal momentum

$$\eta = \frac{15}{32\sqrt{2}} (mT)^{3/2}$$

Bruun (2005)
**Low T: Phonons and Rotons** Effective lagrangian

\[ \mathcal{L} = \varphi^* (\partial_0^2 - v^2) \varphi + i\lambda \dot{\varphi} (\vec{\nabla} \varphi)^2 + \ldots \]

\[ + \varphi^*_{R,v} (i\partial_0 - \Delta) \varphi_{R,v} + c_0 (\varphi^*_{R,v} \varphi_{R,v})^2 + \ldots \]

Shear viscosity

\[ \eta = \eta_R + \frac{c_{Rph}}{\sqrt{T}} e^{\Delta/T} + \frac{c_{ph}}{T^5} + \ldots \]

**High T: Atoms** Viscosity governed by hard core \((V \sim 1/r^{12})\)

\[ \eta = \eta_0 (T/T_0)^{2/3} \]
**Low T: Pions** Chiral perturbation theory

\[ \mathcal{L} = \frac{f_\pi^2}{4} \text{Tr}[\partial_\mu U \partial^\mu U^\dagger] + (B\text{Tr}[MU] + \text{h.c.}) + \ldots \]

Viscosity dominated by $\pi\pi$ scattering

\[ \eta = A \frac{f_\pi^4}{T} \]

**High T: Quasi-Particles** HTL theory (screening, damping, \ldots)

\[ \mathcal{L}_{HTL} = \int d\Omega \, G^a_{\mu\alpha} \frac{v^\alpha v_\beta}{(v \cdot D)^2} G^{a,\mu\beta} \]

Viscosity dominated by t-channel gluon exchange

\[ \eta = \frac{27.13T^3}{g^4 \log(2.7/g)} \]

AMY (2003)
I. Experiment (Liquid Helium)

Kapitza (1938)
viscosity vanishes below $T_c$
capillary flow viscometer

Hollis-Hallett (1955)
roton minimum, phonon rise
rotation viscometer

$\eta/s \simeq 0.8 \hbar/k_B$
II. Scaling Flows (Cold Gases)

transverse expansion  expansion (rotating trap)  collective modes

\[
\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{v}) = 0
\]

\[
mn \frac{\partial \vec{v}}{\partial t} + mn \left( \vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\vec{\nabla} P - n\vec{\nabla} V
\]
Scaling Flows

Universal equation of state

\[ P = \frac{n^{5/3}}{m} f \left( \frac{mT}{n^{2/3}} \right) \]

Equilibrium density profile

\[ n_0(x) = n(\mu(x), T) \quad \mu(x) = \mu_0 \left( 1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} - \frac{z^2}{R_z^2} \right) \]

Scaling Flow: Stretch and rotate profile

\[ \mu_0 \rightarrow \mu_0(t), \quad T \rightarrow T_0(\mu_0(t)/\mu_0), \quad R_x \rightarrow R_x(t), \ldots \]

Linear velocity profile

\[ \vec{v}(x, t) = (\alpha_x x + (\alpha - \omega)y, \alpha_y y + (\alpha + \omega)y, \alpha_z z) \]

“Hubble flow”
Dissipation (Scaling Flows)

Energy dissipation ($\eta, \zeta, \kappa$: shear, bulk viscosity, heat conductivity)

\[
\dot{E} = -\frac{1}{2} \int d^3 x \eta(x) \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right)^2 \\
- \int d^3 x \zeta(x) (\partial_i v_i)^2 - \frac{1}{T} \int d^3 x \kappa(x) (\partial_i T)^2
\]

Have $\zeta = 0$ and $T(x) = \text{const}$. Universality implies

\[
\eta(x) = s(x) \alpha_s \left( \frac{T}{\mu(x)} \right)
\]

\[
\int d^3 x \eta(x) = S\langle \alpha_s \rangle
\]
Navier-Stokes equation

Option 1: Moment method

\[
\int d^3 x \, x_k \left( \rho \dot{v}_i + \ldots \right) = \int d^3 x \, x_k \left( -\nabla_i P - \nabla_j \delta \Pi_{ij} \right)
\]

Only involves \( \langle \eta \rangle/E_0 \).

Option 2: Scaling ansatz for \( \eta(\mu, T) \)

\[
\eta(n, T) = \eta_0 (mT)^{3/2} + \eta_1 \frac{P(n, T)}{T}
\]

Option 3: Numerical solutions.
Dissipation

\[ R_i [\mu m] \]

\[ R_{\perp} \]

\[ R_z \]

\[ \theta [^\circ] \]

\[ E/E_F = 0.56 \]

\[ E/E_F = 2.1 \]

Dissipation

\[
\begin{align*}
(\delta t_0)/t_0 & = \begin{cases} 0.008 \\ 0.024 \end{cases} \left( \frac{\langle \alpha_s \rangle}{1/(4\pi)} \right) \left( \frac{2 \cdot 10^5}{N} \right)^{1/3} \left( \frac{S/N}{2.3} \right) \left( \frac{0.85}{E_0/E_F} \right)
\end{align*}
\]

\[ t_0: \text{“Crossing time” (} b_\perp = b_z, \theta = 45^\circ \text{)} \]

\[ a: \text{amplitude} \]
Time Scales

dissipative
hydro/free streaming
ballistic

\begin{align*}
R_i & [\mu m] \\
R_{\perp} & \\
R_{\|} &
\end{align*}

\[ t [ms] \]
Collective modes: Small viscous correction exponentiates

\[ a(t) = a_0 \cos(\omega t) \exp(-\Gamma t) \]

\[ \langle \eta/s \rangle = (3N\lambda)^{1/3} \left( \frac{\Gamma}{\omega_\perp} \right) \left( \frac{E_0}{E_F} \right) \left( \frac{N}{S} \right) \]

Kinast et al. (2006), Schaefer (2007)
Limitations of Scaling Flows

Simple model for $\eta(n, T)$

$$\eta(n, T) = \eta_0 (mT)^{3/2} + \eta_1 \frac{P(n, T)}{T} f$$

Find exact scaling solutions of the Navier Stokes equation

But: $\eta_0$ completely unconstrained by data

$$\nabla_j [\eta_0 (mT)^{3/2} (\nabla_i v_j + ...)] = 0$$
Relaxation Time Model

In real systems stress tensor does not relax to Navier-Stokes form instantaneously. Consider

\[ \tau_R \left( \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \delta \Pi_{ij} = \delta \Pi_{ij} - \eta \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial \cdot \vec{v} \right) \]

In kinetic theory \( \tau_R \simeq (\eta/n) T^{-1} \)

- dissipation from \( \eta \sim (mT)^{3/2} \): corona excerts drag force.
- modified \( T \) dependence
- modified \( N \) scaling
Where are we?

- high temperature ($T > 2.5T_c$) dominated by corona
- low temperature ($T \sim T_c$): evidence for low viscosity ($\eta/s < 0.4$) core
- also seen in “irrotational flow” data
- full (2nd order hydro or hydro+kin) analysis needed
III. Elliptic Flow (QGP)

Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy.

\[ b \]

\[ \text{Anisotropy Parameter } v_2 \]

\[ \begin{align*}
\text{Hydro model} & \quad \text{PHENIX Data} & \quad \text{STAR Data} \\
\ldots & \quad \pi^+ + \pi^- & \quad \Delta K_S^0 \\
\ldots & \quad K^+ + K^- & \quad \Delta \Lambda \\
\ldots & \quad p + \bar{p} & \quad \Lambda + \bar{\Lambda}
\end{align*} \]

Viscosity and Elliptic Flow

Consistency condition $T_{\mu\nu} \gg \delta T_{\mu\nu}$
(applicability of Navier-Stokes)

$$\frac{\eta + \frac{4}{3} \zeta}{s} \ll \frac{3}{4} (\tau T)$$

Danielewicz, Gyulassy (1985)

Very restrictive for $\tau < 1$ fm

Many questions: Dependence on initial conditions, freeze out, etc.
Outlook

Too early to declare a winner.

\[ \eta/s \simeq 0.8 \text{ (He)}, \quad \eta/s \leq 0.5 \text{ (CA)}, \quad \eta/s \leq 0.5 \text{ (QGP)} \]

Other experimental constraints, more analysis needed.

Kinetic theory: o.k. in He (all \( T \)), o.k. close to \( T_c \) in CA, QGP?

New theory tools: AdS/Cold Atom correspondence? Field theory approaches in cross over regime (large N, epsilon expansions, . . .)