In this homework we consider the Lorentz group. The defining representation is four dimensional. The Lorentz group is the set of real matrices $\Lambda$ that leave the Minkowski space metric invariant

$$\Lambda^T \eta \Lambda = \eta, \quad \eta = \text{diag}(1, -1, -1, -1), \quad \text{det}(\Lambda) = 1.$$ 

With this definition the Lorentz group preserves the inner product of 4-vectors. This means that $v_\mu w^\mu = \eta_{\mu\nu} v^\nu w^\mu = v^T \eta w$ is invariant under $v \to \Lambda v$, $w \to \Lambda w$.

1. If we write $\Lambda = \exp(i\alpha_a X_a)$ show that the generators $X_a$ must satisfy

$$X_a^T \eta + \eta X_a = 0.$$ 

Determine from this condition and the reality of $\Lambda$ the number of generators.

2. Show that the six matrices

$$(J_i)_{\mu\nu} = -i\epsilon_{0i\mu\nu}, \quad (K_i)_{\mu\nu} = -i(\delta_{\mu0}\delta_{\nu i} - \delta_{\mu i}\delta_{\nu0}),$$

with $i = 1, 2, 3$ and $\mu, \nu = 1, \ldots, 4$ form a complete basis of the generators $X_a$. Here, $\epsilon_{\mu\nu\alpha\beta}$ is the completely anti-symmetric four index tensor with $\epsilon_{0123} = +1$. The $J_i$ and $K_i$ are called rotations and boosts, respectively. Are $J_i, K_i$ hermitean?


4. Define $A_i = (J_i + iK_i)/2$ and $B_i = (J_i - iK_i)/2$. Compute $[A_i, A_j]$, $[B_i, B_j]$, $[A_i, B_j]$. Your result shows that the algebra of the Lorentz group is isomorphic to that of $SU(2) \times SU(2)$. This implies that representations of the Lorentz group are labeled by pairs of half-integers $(j_A, j_B)$.

5. Using

$$\alpha_a X_a = \theta_i J_i + \omega_i K_i = (\theta_i - i\omega_i)A_i + (\theta_i - i\omega_i)B_i$$

you can find the explicit form of a Lorentz transformations in the $(j_A, j_B)$ representation. As an example, consider a two-spinor $\psi = (\alpha, \beta)^T$ in the $(1/2, 0)$ representation. How does $\psi$ transform under rotations around the $z$-axis by an angle $\theta_3$ or boosts along the $z$-axis by a boost parameter $\omega_3$?