Chapter 34 #36) What is KE of a proton whose wavelength is 1 nm and 1 fm?

The deBroglie relation gives us the momentum of the proton, \( \mathbf{p} = \frac{\hbar}{\lambda} \).

For a nonrelativistic particle, \( KE = \frac{p^2}{2m} \), so

\[
\lambda = 1 \text{ nm} \quad KE = \frac{\hbar^2}{\lambda^2 m} = \left( \frac{\hbar c}{\lambda} \right)^2 \frac{1}{2mc^2}
\]

\[
= \left( \frac{1240 \text{ eV nm}}{1 \text{ nm}} \right)^2 \frac{1}{2 \times 938 \times 10^6 \text{ eV}} = 8.2 \times 10^{-4} \text{ eV}
\]

\[
\lambda = 1 \text{ fm} \quad KE = \left( \frac{1240 \text{ eV nm}}{10^{-6} \text{ nm}} \right)^2 \frac{1}{2 \times 938 \times 10^6 \text{ eV}} = 8.2 \times 10^8 \text{ eV}
\]

This is \( \sim mc^2 \), so we need to use the relativistic relationship for total energy \( E = \sqrt{p^2 c^2 + mc^4} \)

\[
E = \sqrt{\left( \frac{\hbar c}{\lambda} \right)^2 + (mc^2)^2} = \sqrt{\left( \frac{(1240 \text{ eV nm})^2}{10^{-6} \text{ nm}} \right) + (9.38 \times 10^6 \text{ eV})^2}
\]

\[
= 1.55 \times 10^9 \text{ eV}
\]

\[
KE = E - mc^2 = 1.55 \times 10^9 \text{ eV} - 9.38 \times 10^6 \text{ eV}
\]

\[
KE = 617 \text{ MeV}
\]
#42) A neutron is in a 1D box of width \( L = 0.2 \) nm.

a) Find the energy of the ground state and first two excited states.

The allowed energies of a particle in a 1D box are given by eqn 34-21
in the text:

\[
E_n = n^2 \left( \frac{(hc)^2}{mec^2} \right) \frac{1}{8L^2}
\]

\[
= n^2 \left( \frac{1240 \text{eV nm}}{939 \times 10^4 \text{eV}} \right)^2 \frac{1}{8 \times (0.2 \text{nm})^2} = 5.1 \times 10^{-3} \text{eV} \times n^2
\]

Ground state, \( n=1 \), \( E_1 = 5.1 \times 10^{-3} \text{eV} \)

First excited state, \( n=2 \), \( E_2 = 2.0 \times 10^{-2} \text{eV} \)

Second excited state, \( n=3 \), \( E_3 = 4.6 \times 10^{-2} \text{eV} \)

b) What is \( \lambda \) of photon emitted when the neutron transitions from \( n=2 \) to \( n=1 \)?

The energy of the photon, \( E = \frac{hc}{\lambda} \), will equal the difference in energy between the two states:

\[
\frac{hc}{\lambda} = E_2 - E_1 = 2.0 \times 10^{-2} \text{eV} - 5.1 \times 10^{-3} \text{eV}
\]

\[
\lambda = \frac{1240 \text{eV nm}}{14.9 \times 10^{-3} \text{eV}} = 8.3 \times 10^4 \text{nm} = 83 \mu\text{m}
\]

c) For \( n=3 \rightarrow 2 \), \( \Delta E = 2.6 \times 10^{-2} \text{eV} \), \( \lambda = 48 \mu\text{m} \)

d) For \( n=3 \rightarrow 1 \), \( \Delta E = 4.1 \times 10^{-2} \text{eV} \), \( \lambda = 30 \mu\text{m} \)
#43) particle in a box

The ground state is given by the wave function

\[ \psi_1(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{\pi x}{L} \right) \]  

eqn 34-25

Find the probability of finding the particle in a narrow interval \( \Delta x = 0.002 L \) and centered at \( x = \frac{1}{4} L, \frac{1}{2} L, \frac{3}{4} L \)

The probability is given by \( P(x)dx = \psi^2 dx \)

\[ P = \frac{2}{L} \sin^2 \left( \frac{\pi x}{4L} \right) (0.002) = (0.004) \sin^2 \left( \frac{\pi}{4} \right) \]

\[ P = 0.002 \]

At \( x = \frac{1}{2} L \), \( P = \frac{2}{L} (0.002) \sin^2 \left( \frac{\pi}{2} \right) = 0.004 \)

At \( x = \frac{3}{4} L \), symmetry suggests the probability should be the same as at \( x = \frac{1}{4} L \)

\[ P = \left( \frac{2}{L} \right) (0.002) \sin^2 \left( \frac{\pi}{4L} \right) = 0.002 \]
#46) particle in a box, in the first excited state.

\[ y = \sqrt{\frac{\lambda}{L}} \sin \left( \frac{2\pi x}{L} \right) \]

![Graph of a sine function]

Expectation value \( \langle x \rangle = \int_0^L x^2 y^2 \, dx \)

By symmetry, we expect the average measured position to be \( \frac{L}{2} \).

\[ \langle x \rangle = \frac{L}{2} \int_0^L x^2 \sin^2 \left( \frac{2\pi x}{L} \right) \, dx = \frac{L^2}{4\pi^2} \int_0^{2\pi} u \sin^2 u \, du \]

\[ \int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x \]

\[ \langle x \rangle = \frac{L^2}{2\pi^2} \left\{ \frac{(2\pi)^2}{4} - \frac{1}{8} \cos 4\pi + \frac{1}{8} \cos 0 \right\} \]

\[ = \frac{L^2}{2\pi^2} \pi^2 = \frac{L}{2} \checkmark \]

The probability of finding the particle at \( x = \frac{L}{2} \) in a narrow region of width \( dx \) is given by

\[ P_{\text{dx}} \, dx = y^2 \, dx = 0 \quad \text{since} \quad y \left( \frac{L}{2} \right) = 0. \]

This is not inconsistent. \( \langle x \rangle = \frac{L}{2} \) means one is just as likely to find the particle to the left of \( \frac{L}{2} \) as the right, but nothing about the particle at \( \frac{L}{2} \).
A 1D box centered at the origin

\[
\begin{array}{c}
  \vdots \\
  -\frac{L}{2} & \frac{L}{2} & \frac{L}{2} \\
  \vdots
\end{array}
\]

The wave function of the ground state is given by

\[\psi_1(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right)\]

The expectation value of \(x\) is given by

\[\langle x \rangle = \int_{-\frac{L}{2}}^{\frac{L}{2}} x \psi_1^2(x) dx\]

\[\langle x \rangle = \frac{L}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} x \cos^2\left(\frac{\pi x}{L}\right) dx\]

\[\langle x \rangle = \frac{L}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} x \cos^2\left(\frac{\pi x}{L}\right) dx\]

\[= \frac{L}{2} \left[ \frac{x^2}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos^2\left(\frac{\pi x}{L}\right) dx \right]
\]

\[= \frac{L}{2} \left[ \frac{x^2}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1 + \cos(2u)}{2} du \right]
\]

\[= \frac{L}{2} \left[ \frac{x^2}{2} \left. \left( \frac{u}{4} + \frac{1}{4} \sin(2u) + \frac{1}{8} \cos(2u) \right) \right|_{-\frac{L}{2}}^{\frac{L}{2}} \right]
\]

\[= \frac{L}{2} \left[ \frac{x^2}{2} \left( \frac{L^2}{16} - \frac{L^2}{16} + \frac{1}{8}(-1) - \frac{1}{8}(-1) \right) \right]
\]

\[= 0 \Rightarrow \langle x \rangle = 0
\]

\[\langle x^2 \rangle = \frac{L}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \cos^2\left(\frac{\pi x}{L}\right) dx
\]

\[= \frac{L}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \cos^2\left(\frac{\pi x}{L}\right) dx
\]

\[= \frac{L}{2} \left[ \frac{x^3}{3} \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos^2\left(\frac{\pi x}{L}\right) dx \right]
\]

\[= \frac{L}{2} \left[ \frac{x^3}{3} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1 + \cos(2u)}{2} du \right]
\]

\[= \frac{L}{2} \left[ \frac{x^3}{3} \left. \left( \frac{u}{4} + \frac{1}{4} \sin(2u) + \frac{1}{8} \cos(2u) \right) \right|_{-\frac{L}{2}}^{\frac{L}{2}} \right]
\]

\[= \frac{L}{2} \left[ \frac{x^3}{3} \left( \frac{L^3}{16} + \frac{L^3}{16} + \frac{1}{8}(-1) - \frac{1}{8}(-1) \right) \right]
\]

\[= \frac{L^3}{12}\left( 1 + 1 - \frac{1}{8} + \frac{1}{8} \right)
\]

\[\langle x^2 \rangle = \frac{L^3}{12}\left( \frac{1}{2} - \frac{3\pi^2}{4} \right)
\]