Relativity in Action

David Brown (Cornell, 2004)
Jimmy's papers on Black Hole Thermodynamics


“Quasilocal energy in general relativity” in Mathematical Aspects of Classical Field Theory edited by M.J. Gotay, J.E. Marsden, and V. Moncrief (American Mathematical Society, Providence, 1992) 129-142. (with J.D. Brown)


Jimmy's papers on Black Hole Thermodynamics


Change of boundary terms/boundary conditions in the action

 Canonical transformation (dynamical level)

 Legendre transformation (thermodynamical level)

Gravity is the essential ingredient!
...this talk...

- Initial value problem
- Formulations of Einstein's equations
- Numerical relativity

Action
Usual approach to numerical modeling:

Continuum Eqns of Motion

\[ \text{Discretize} \]

Discrete Eqns of Motion
Usual approach to numerical modeling:

Action

\[ \text{Vary} \]

Continuum Eqns of Motion

\[ \text{Discretize} \]

Discrete Eqns of Motion
Usual approach to numerical modeling:

- Action
  - Vary
  - Continuum Eqns of Motion
  - Discretize
  - Discrete Eqns of Motion

Variational Integrator approach to numerical modeling*:

- Action
  - Discretize
  - Discrete Action
  - Vary
  - Discrete Eqns of Motion

*Kane, Marsden, Ortiz, Patrick, Pekarsky, Shkoller, West
WHY BOTHER?

The action principle provides a unique perspective that can lead to important insights into the physical and mathematical descriptions of a dynamical system.
WHY BOTHER?

• The action principle provides a unique perspective that can lead to important insights into the physical and mathematical descriptions of a dynamical system.

• Variational integrators typically do a superior job of conserving energy.
Hamiltonian mechanics

\[ S[p, x] = \int_{t_1}^{t_2} dt \ [p \dot{x} - H(x, p)] \]

zone centers: \( n = \)

\[ \begin{array}{cccc}
  1 & 2 & 3 \\
\end{array} \]

nodes: \( n = \)

\[ \begin{array}{cccc}
  0 & 1 & 2 & 3 \\
\end{array} \]

\[ S[p, x] = \sum_{n=1}^{N} \Delta t \left[ p^n \left( \frac{x^n - x^{n-1}}{\Delta t} \right) - H(x^n, p^n) \right] \]

\[ x^n = \frac{x^n + x^{n-1}}{2} \]
Variation of the action

\[ \delta S = \sum_{n=1}^{N} \Delta t \left[ \frac{x^n - x^{n-1}}{\Delta t} - \left( \frac{\partial H}{\partial p} \right)^n \right] \delta p^n + \sum_{n=1}^{N-1} \Delta t \left[ \frac{p^n - p^{n+1}}{\Delta t} - \left( \frac{\partial H}{\partial x} \right)^{n+1} \right] \delta x^n \]

\[ + \left[ p^N - \frac{\Delta t}{2} \left( \frac{\partial H}{\partial x} \right)^N \right] \delta x^N - \left[ p^1 + \frac{\Delta t}{2} \left( \frac{\partial H}{\partial x} \right)^1 \right] \delta x^0 \]

implies

\[ \frac{x^{n+1} - x^n}{\Delta t} = \left( \frac{\partial H}{\partial p} \right)^{n+1}, \quad n = 0, \ldots, N - 1 \]

\[ \frac{p^{n+1} - p^n}{\Delta t} = -\left( \frac{\partial H}{\partial x} \right)^{n+1}, \quad n = 1, \ldots, N - 1 \]

with \( x^0 \) and \( x^N \) fixed.
Example: Harmonic oscillators with nonlinear coupling
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Why is energy conservation so important?
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Because energy conservation in mechanics is analogous to the preservation of constraints in general relativity. This is seen most clearly with the parametrized form of the action...

\[
S[p, x, t, \pi, \alpha] = \int_{\sigma_1}^{\sigma_2} d\sigma \ [p\dot{x} + \pi\dot{t} - \alpha \mathcal{H}]
\]

\[
\mathcal{H} \equiv \pi + H(x, p) = \text{Hamiltonian constraint}
\]

\[
\alpha = \text{lapse function}
\]
Preservation of the constraints in numerical relativity is crucial. As documented by the Cornell/CalTech group, the run time for “free evolution” codes is limited by unphysical, exponentially growing, constraint violating modes.

Variational integrators are good at staying close to the “constraint hypersurface”, at least for mechanical systems.
Look again at the parametrized form of the action...

\[ S[p, x, t, \pi, \alpha] = \int_{\sigma_1}^{\sigma_2} d\sigma \ [p\dot{x} + \pi \dot{t} - \alpha \mathcal{H}] \]

\[ \mathcal{H} \equiv \pi + H(x, p) = \text{Hamiltonian constraint} \]

\[ \alpha = \text{lapse function} \]

This system, like GR, is both underdetermined (because the equations of motion don't determine the lapse) and overdetermined (because the variables that are determined by the equations of motion are also constrained). ---JWY

What if we apply the variational integrator construction to this action?
VI equations of motion:

\[
\frac{x^{n+1} - x^n}{\Delta \sigma} = \left( \alpha \frac{\partial \mathcal{H}}{\partial p} \right)^{n+1}, \quad n = 0, \ldots, N - 1
\]

\[
\frac{t^{n+1} - t^n}{\Delta \sigma} = \alpha^{n+1}, \quad n = 0, \ldots, N - 1
\]

\[
\mathcal{H}^{n+1} = 0, \quad n = 0, \ldots, N - 1
\]

\[
\frac{p^{n+1} - p^n}{\Delta \sigma} = - \left( \alpha \frac{\partial \mathcal{H}}{\partial x} \right)^{n+1}, \quad n = 1, \ldots, N - 1
\]

\[
\frac{\pi^{n+1} - \pi^n}{\Delta \sigma} = 0, \quad n = 1, \ldots, N - 1
\]

The “dot” equations do not preserve the value of the constraint. The system is not overdetermined/underdetermined! Five equations for five unknowns, with the constraint explicitly enforced at each timestep.
For the coupled oscillators...

The time step adjusts itself during the evolution to maintain the constraint.
For the coupled oscillators...
By discretizing the action for a constrained Hamiltonian system, then varying that action, we obtain discrete equations of motion that explicitly maintain the constraints. The price we pay is a loss of freedom to choose the evolution for the constraint multipliers.

The undetermined multipliers of the continuum theory become Lagrange (determined) multipliers in the discrete theory.
Is this price too high to pay for numerical relativity?

Not necessarily. If needed, we can always “guide” the time slices by making occasional adjustments using the prior spacetime solution:
Let's boldly forge ahead and apply the VI construction to general relativity.

Some issues I want to avoid for now:

- What is the preferred form of the action/equations of motion at the continuum level? Can we write an action for the Einstein-Christoffel system? BSSN? (Answer is yes, with some caveats...)

- How do we implement spatial boundary conditions that are consistent with the variational principle, and physically appropriate for radiating systems?
Consider the ADM action with periodic BC's:

\[
S[g_{ab}, P^{ab}] = \int dt \, d^3x \left[ P^{ab} \dot{g}_{ab} - \alpha \mathcal{H} - \beta^a \mathcal{M}_a \right] 
\]

\[
\mathcal{H} = 2P^{ab}P_{ij} - P^2 - gR/2 
\]

\[
\mathcal{M}_a = -2D_b P^b_a 
\]

The VI equations have the form:

\[
\frac{\delta S}{\delta (P^{ab})^{n+1}_{ijk}} = 0, \quad n = 0, \ldots, N - 1 
\]

\[
(\mathcal{H})^{n+1}_{ijk} = 0, \quad n = 0, \ldots, N - 1 
\]

\[
(\mathcal{M}_a)^{n+1}_{ijk} = 0, \quad n = 0, \ldots, N - 1 
\]

\[
\frac{\delta S}{\delta (g_{ab})^n_{ijk}} = 0, \quad n = 1, \ldots, N - 1 
\]

to be solved for \( (g_{ab})^n_{ijk}, \ (P^{ab})^n_{ijk}, \ (\alpha)^n_{ijk}, \) and \( (\beta^a)^n_{ijk} \)
The form of the equations depends on the choice of whether the coordinates, momenta, and multipliers are cell centered or node centered in space and in time. In practice this choice is very important!

The equations are implicit, making them a challenge to solve.

The equation obtained by varying the metric ranges from $n = 1,...,N-1$, while the other equations range from $n = 0,...,N-1$. More on this later.
Example: GR with plane symmetry described by the model*

\[
g_{ab} = \begin{pmatrix}
g(x) & 0 & 0 \\
0 & h(x) & 0 \\
0 & 0 & h(x)
\end{pmatrix}
\]

\[
P^{ab} = \begin{pmatrix}
P(x) & 0 & 0 \\
0 & Q(x) & 0 \\
0 & 0 & Q(x)
\end{pmatrix}
\]

\[
S[g, h, P, Q, \alpha, \beta] = \int dt \, dx \left( P \dot{g} + 2Q \dot{h} - \alpha H - \beta M \right)
\]

\[
H = \frac{1}{\sqrt{gh}} \left[ P^2 g^2 - 4PQgh + hh'' - \frac{hh'g'}{2g} - \frac{(h')^2}{4} \right]
\]

\[
M = -2gP' - g'P + 2h'Q
\]

*Solutions are cosmologies with a “big crunch”.*
Metric function $g(x)$ after 0th and fifth timesteps
Lapse function at the second timestep:

Lapse at 5, 10, 15,...,40 timesteps:
Log(L2 norm of Hamiltonian constraint) versus time for solutions of varying numerical accuracy.
However...we can't solve for the shift vector (i.e. we can't solve the momentum constraint). Let's take a closer look:
However...we can't solve for the shift vector (i.e. we can't solve the momentum constraint). Let's take a closer look:

We want to solve the constraints at each timestep. In particular for the initial timestep (remember the n=0 equations), we need to solve

\[
\frac{\delta S}{\delta (P_{ij}^{ab})} = 0, \\
(\mathcal{H})_{ijk}^{1} = 0, \\
(\mathcal{M}_a)_{ijk}^{1} = 0.
\]

This is a discrete version of the original “thin sandwich” problem of Baierlein, Sharp and Wheeler: Given the metric on two nearby time slices, and the definition for the extrinsic curvature, solve the Hamiltonian and momentum constraints for the lapse and shift.

Sometimes it works, but not generically!
We need to make the thin sandwich problem well posed. Jimmy has shown us how to do it with his conformal thin sandwich construction:* 

- split the metric into a conformal factor and a background 
- split the extrinsic curvature (momentum) into its trace and trace-free parts 
- add conformal weights,... 

\[
g_{ab} \equiv \phi^4 \tilde{g}_{ab} \quad \text{with } \sqrt{\tilde{g}} \text{ constant in time}
\]

\[
P^{ab} \equiv \frac{1}{2} \phi^2 \tilde{g}^{ab} \tau + \phi^{-4} \tilde{A}^{ab} \quad \text{with } \tilde{A}^{ab} \text{ trace free}
\]

implies

\[
P^{ab} \dot{g}_{ab} = 6\phi^5 \tau \phi + \tilde{A}^{ab} \tilde{g}_{ab}
\]

*Phys. Rev. Lett. 82 (1999) 1350
Throw in a canonical transformation to make “tau” a coordinate with “-phi^6” its conjugate momentum...

\[
S[\phi, \tilde{g}, \tau, \tilde{A}] = \int dt \ d^3 x \left( -\phi^6 \dot{\tau} + \tilde{A}^{ab} \tilde{g}_{ab} - \alpha \mathcal{H} - \beta^a \mathcal{M}_a \right)
\]

\[
\mathcal{H} = 4\tilde{g} \phi^7 \tilde{D}^2 \phi - \frac{1}{2} \tilde{g} \phi^8 \tilde{R} - \frac{3}{4} \phi^{12} \tau^2 + 2\tilde{A}^{ab} \tilde{A}_{ab}
\]
\[
\mathcal{M}_a = -2\tilde{D}_b \tilde{A}^b_a - \phi^6 \tilde{D}_a \tau
\]

Now discretize the action. The initial timestep (n = 0) equations are discrete versions of

\[
\frac{\delta S}{\delta \tilde{A}^{ab}} = 0 ,
\]
\[
\frac{\delta S}{\delta \phi} = 0 ,
\]
\[
\mathcal{H} = 0 ,
\]
\[
\mathcal{M}_a = 0
\]
These are the extended conformal thin sandwich equations! (Pfeiffer and York, PRD67 (2003) 044022)

\[
\tilde{A}_{ab} = \frac{1}{4 \alpha} \left[ \dot{\tilde{g}}_{ab} - 2 \tilde{D}_{(a} \beta_{b)} + \frac{2}{3} \tilde{g}_{ab} \tilde{D}_{c} \beta^{c} \right]
\]

\[
\tilde{D}^{a} \tilde{D}_{a} \alpha = \alpha \left[ \frac{7}{\tilde{g} \phi^{8}} \tilde{A}^{ab} \tilde{A}_{ab} - \frac{3}{4} \tilde{R} - \frac{3 \phi^{4}}{8 \tilde{g}} \tau^{2} - 42 (\tilde{D}^{a} \ln \phi) (\tilde{D}_{a} \ln \phi) \right]
+ 14 (\tilde{D}^{a} \alpha) (\tilde{D}_{a} \ln \phi) + \frac{3}{2 \tilde{g} \phi^{2}} (\dot{\tau} - \beta^{a} \tilde{D}_{a} \tau)
\]

\[
\tilde{D}^{a} \tilde{D}_{a} \phi = \frac{\phi}{8} \tilde{R} + \frac{3 \phi^{5}}{8 \tilde{g}} \tau^{2} - \frac{1}{2 \tilde{g} \phi^{7}} \tilde{A}^{ab} \tilde{A}_{ab}
\]

\[
(\tilde{\Delta}_{L} \beta)_{a} = (\tilde{\mathcal{L}} \beta)^{b}_{a} (\tilde{D}_{b} \ln \alpha) - \alpha \tilde{D}^{b} (\dot{\tilde{g}}_{ab} / \alpha) - 2 \alpha \phi^{6} \tilde{D}_{a} \tau
\]

where

\[
(\tilde{\Delta}_{L} \beta)_{a} \equiv \tilde{D}^{b} (\tilde{\mathcal{L}} \beta)_{ab} \quad \text{and} \quad (\tilde{\mathcal{L}} \beta)_{ab} \equiv 2 \tilde{D}_{(a} \beta_{b)} - \frac{2}{3} \tilde{g}_{ab} \tilde{D}_{c} \beta^{c}
\]
The essential content of the extended conformal thin sandwich equations is a set of five nice, well behaved elliptic equations for the lapse, shift, and conformal factor. Generically, we expect solutions to exist.

The discrete action principle directs us to choose

\[ \tilde{g}_{ab}, \: \tilde{g}_{ab}, \: \tau, \: \text{and} \: \tilde{r} \]

as free initial data. The “level \( n = 0 \)” equations are the conformal thin sandwich equations in extended form. These determine the remaining initial data

\[ \tilde{A}^{ab}, \: \phi, \: \alpha, \: \text{and} \: \beta^a \]
Question: can we always solve the $n = 1, 2, \ldots$ equations for the coordinates, momenta, and multipliers?

Answer: I don't know. There is no continuum version of this question.
Question: Is the loss of freedom to choose the lapse and shift too high of a price to pay? If so...

Option 1: As mentioned before, guide the slices by using the spacetime solution to construct new “initial” data during the course of the evolution.

Option 2: Don't solve the constraints. The VI discretization of the evolution equations might do a good enough job of staying close to the constraint hypersurface anyway. (Recall the first example of the coupled oscillators.)
Question: Is the loss of freedom to choose the lapse and shift too high to pay? If so...

Option 3: Apply a “studder-step” evolution:

- Specify the lapse and shift
- Solve the evolution equations for the canonical coordinates and momenta at the next timestep
- Solve the conformal thin sandwich equations using the canonical coordinates at the two neighboring timesteps

This procedure is well defined at each timestep and keeps the constraints satisfied. The lapse and shift are freely specified but then subject to a (perhaps small) correction. The price we pay is that the discrete evolution equations obtained by varying the action with respect to the coordinates (the “momentum-dot” equations) are not satisfied exactly.