Homework 12, due 12-9

1. The Lagrange function for a point particle with mass $m$ and charge $-e$ in an electromagnetic field is

$$ L = \frac{m}{2} \dot{\vec{v}}^2 + e\phi - \frac{e}{c} \dot{\vec{v}} \cdot \vec{A}, $$

where $\vec{v} = \dot{\vec{x}}$ is the velocity, $\phi$ is the scalar potential and $\vec{A}$ is the vector potential. Show that the Hamilton function $\mathcal{H}$ is given by

$$ \mathcal{H} = \frac{1}{2m} \left( \vec{p} + \frac{e}{c} \vec{A} \right)^2 - e\phi. $$

2. Consider a particle of charge $-e$ in a vector potential

$$ \vec{A} = \frac{B}{2} (-y \hat{e}_x + x \hat{e}_y), $$

where $\hat{e}_x$ and $\hat{e}_y$ are unit vectors in the $x, y$ direction. Show that the magnetic field is $\vec{B} = B \hat{e}_z$. Also show that a classical particle in this potential moves in circles with an angular frequency $\omega_0 = eB/(mc)$.

3. Consider the corresponding quantum Hamiltonian

$$ H = \frac{1}{2m} \left\{ \left( p_x - \frac{e}{2c} yB \right)^2 + \left( p_y + \frac{e}{2c} xB \right)^2 + p_z^2 \right\}. $$

Find the spectrum of $H$. Hint: Introduce the operators

$$ Q = -\frac{1}{eB} \left( cp_x - \frac{e}{2} yB \right), \quad P = \left( p_y + \frac{e}{2c} xB \right). $$

Compute $[P, Q]$ and express $H$ in terms of $P, Q$.

4. Compute the ground state wave function. Hint: Follow the strategy we employed in the case of the harmonic oscillator. Introduce the complex variables $u = x + iy$ and $u^* = x - iy$. Use the ansatz

$$ \psi_0 = f(z)g(u, u^*) \exp(-\alpha |u|^2), $$

with a suitably chosen $\alpha$. Is the solution for $g$ unique? Explain!